Math Prize Competition!

Western Michigan University

October 7, 2017

Name (printed legibly): ______________________________

Wmich email address: ______________________________

Read all of the following before working on the contest problems:

- This contest is **closed book**. Discussing the problems with colleagues is **not** permitted during the contest. The use of a calculator is **not** allowed for this contest. The use of a cell phone is **not** allowed for this contest. A violation of these rules results in immediate disqualification.
- Show or explain all of your work. Please write clearly and completely. Unjustified answers are regarded as incorrect.
- Please keep your written answers brief; be clear and to the point. (Do not ramble.)
- The problems are not necessarily in order of difficulty. If you spend more than 5–10 minutes thinking about a problem with no progress, you should probably look at the next one to see if it's easier.
- This contest has 6 problems and each problem is worth 10 points. If you do not have every page, please inform a proctor.
- You have two hours to complete your work on these problems.
- If you need clarification on any problem, please ask a proctor.
- Good luck!

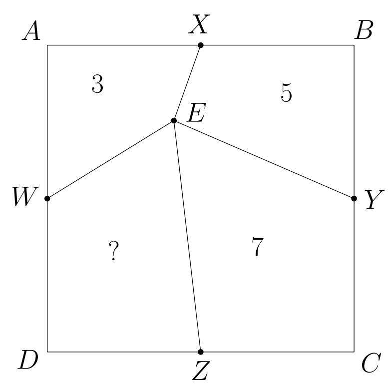
Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total	**60**	

Problem 1. Given a triangle ABC. Let h_a, h_b, h_c be the altitudes to its sides a, b, c, respectively. Prove that

$$\frac{1}{h_a} + \frac{1}{h_b} > \frac{1}{h_c}.$$

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $f(x, y) + f(y, z) + f(z, x) = 0$ for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = g(x) - g(y)$ for all real numbers x and y.

Problem 3. Point E is on the interior of square $ABCD$. Points X, Y, Z and W are the midpoints of sides AB, BC, CD, and DA respectively. The area of quadrilateral $AXEW$ is 3, the area of $BXEY$ is 5, and the area of $CYEZ$ is 7. Find the area of $DZEW$.



Problem 4.

a) Determine all pairs (m, n) of positive integers such that $2^n + 1 = m^2$.

b) Assuming $p > 1$ is prime, determine all pairs (m, n) of positive integers such that $p^n + 1 = m^p$.

Problem 5. Find all real numbers x such that

$$2\sqrt[3]{2x - 1} = x^3 + 1.$$

Problem 6. How many sets $S = \{a, b, c\}$ of three distinct elements of $\{1, 2, 3, ..., 300\}$ are there such that $a + b + c$ is divisible by 3?