Graph Theory
Preliminary Exam

31 May 1997
9:00 a.m. - 3:00 p.m.

Instructions: Do all ten problems.

1. (6 points) Prove that if G is a (connected) graph with diam $G \geq 3$, then the domination number of its complement $\overline{G}$ is 2.

2. (8 points) Let G be a hamiltonian graph of order at least 4 such that $G - v$ is hamiltonian for every vertex v of G. Prove that for every proper subset S of vertices of G with $|S| \geq 2$, $k(G - S) \leq |S| - 1$. [Note: Recall that $k(H)$ denotes the number of components of a graph H.]

3. (8 points) Let T be a tree of order at least 5, and let $e_1, e_2, \cdots, e_5 \in E(\overline{T})$. Let $G = T + \{e_1, e_2, \cdots, e_5\}$. Prove that if G does not contain a subdivision of $K_{3,3}$, then G is planar.

4. (8 points) Let G be a noncomplete graph of order n and connectivity k such that $\deg v \geq \frac{n+kt-t}{t+1}$ for some integer $t \geq 2$. Show that if S is a vertex-cut of cardinality $\kappa(G)$, then $G - S$ has at most t components.

5. (8 points) Let G be a critically k-chromatic graph for some $k \geq 2$ with the property that every proper induced subgraph of G is perfect. Prove that G is perfect if and only if G is complete.

6. (10 points)

   (a) Let G be a graph of order $n \geq 4$. If $\deg v \geq \frac{2n+1}{3}$ for every vertex v of G, then prove that every edge of G belongs to a complete subgraph of order 4.

   (b) Show that the result in (a) is best possible in general by showing that $\frac{2n+1}{3}$ cannot be replaced by $\frac{2n}{3}$.

7. (10 points) A tournament T of order at least 6 has the property that every vertex of T belongs to a 3-cycle but to no cycle of length greater than 4. Show that if u and v are vertices of T such that $\text{od } u = \text{od } v + 1$, then T contains both a directed $u - v$ path and a directed $v - u$ path.

8. (14 points)

   (a) Prove that there exist exactly two (nonisomorphic) 3-regular graphs of order 6.

   (b) Prove that $K_{2,2,2} \rightarrow (K_3, P_3)$.

   (c) Prove that $K_{2,2,2} \not\rightarrow (K_3, K_3)$.

9. (14 points)

   (a) Petersen's theorem states that if G is a bridgeless cubic graph, then G has a 1-factor. Show that Petersen's theorem can be extended somewhat by proving that if G is a bridgeless graph all of whose vertices have degree 3 except one, which has degree 7, then G has a 1-factor.

   (b) Prove that the result in (a) cannot be improved by showing that there exists a bridgeless graph with degree set $\{3, 7\}$ containing exactly two vertices of degree 7 and no 1-factor.

10. (14 points) A graph G is obtained from K_6 by deleting three independent edges. Let $V(G) = \{v_1, v_2, \cdots, v_6\}$. The 6-tuple $(\pi_1, \pi_2, \cdots, \pi_6)$ of cyclic permutations defined by

$$\pi_1 = (3456), \quad \pi_2 = (3465), \quad \pi_3 = (1256),$$
$$\pi_4 = (2165), \quad \pi_5 = (2134), \quad \pi_6 = (1243)$$

describes a 2-cell embedding of G on S_k.

(a) What is k?

(b) Is k the genus of G? Explain.

(c) Is k the maximum genus of G? Explain.

(d) Determine, with explanation, all surfaces on which G can be 2-cell embedded.