Algebra Preliminary Exam

March 3, 2006

1. Let G be a group.

   (a) Suppose that A and B are normal subgroups of G so that $G = AB$ and $A \cap B = \{e\}$. Show that $G \simeq A \times B$.

   (b) Show that if $|G| = p^2$ for some prime p then G is abelian.

   (c) Classify all groups of order 1225.

2. A subgroup H of a group G is called *characteristic* in G if every automorphism of G maps H to itself. The *commutator subgroup* G' of a group G is the group generated by all commutators in G, i.e., by all elements of the form $a^{-1}b^{-1}ab$ with $a, b \in G$. If H is a subgroup of G then the *Focal subgroup* of H with respect to G is $H^* = \{h^{-1}h^g | h \in H, h^g \in H, g \in G\}$. Here $h^g = g^{-1}hg$.

   Let G be a group, and H a subgroup of G. Prove the following statements.

   (a) Every characteristic subgroup is a normal subgroup.

   (b) $G' \lhd G$, and G/G' is abelian.

   (c) H' is a subgroup of H^*.

   (d) $H^* \lhd H$ and H/H^* is abelian.

   (e) If K is a subgroup of G such that $H^* \subseteq K \subseteq H$ then $K \lhd H$ and H/K is abelian.

3. Let R be a PID.

   (a) Show that every nonzero prime ideal of R is maximal.

   (b) If S is an integral domain and $\phi : R \longrightarrow S$ a surjective ring homomorphism, show that either ϕ is an isomorphism or S is a field.

   (c) Show that if $R[x]$ is a PID, then R is a field.

4. (a) Let n be a positive integer. Show that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is prime.

   (b) Construct the 4-element field $\mathbb{F}_4$ by exhibiting its addition and multiplication tables.

5. (a) Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} = \{0\}$ if $\gcd(m, n) = 1$.

   (b) Characterize $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ in general.

   (c) If G is a finite abelian group and $G \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = \{0\}$ for all primes p, show that $G = \{0\}$. Does the result remain true if G is infinite?

6. If M is a finitely generated module over a field K, prove that every submodule of M is free, and that every quotient of M is free.

7. Let E be the splitting field of $x^6 - 17$ over $\mathbb{Q}$.

   (a) Find the Galois group G of E over $\mathbb{Q}$ and identify it with a well-known group.

   (b) Find a subfield F of E such that E has dimension 2 over F, and F is Galois over $\mathbb{Q}$. (Hint: Consider the center of G.).