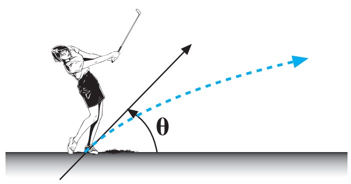
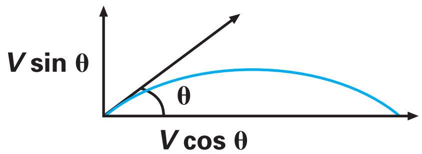
Lesson 2

Reasoning with Trigonometric Functions

In Unit 2, "Modeling Motion," you learned that projectile motion such as the path of a baseball or golf ball is the sum of a horizontal and a vertical component. When the initial velocity is V and the angle of elevation is θ, the initial magnitudes of the components are $V \cos \theta$ and $V \sin \theta$.





Ignoring air resistance, the height (in feet) of the projectile after t seconds is given by $h(t) = (V \sin \theta) \cdot t + (16t \cdot t \sin 270^\circ)$ or $h(t) = (V \sin \theta) \cdot t - 16t^2$. While in the air, the horizontal distance (in feet) the projectile travels in t seconds is given by $d(t) = (V \cos \theta) \cdot t$.

Athletes are often interested in the value of $d(t)$ when $h(t) = 0$. That is, they are interested in how far the baseball or golf ball will travel before it returns to the ground. This distance (in feet), called the *range*, is given by

$$R(\theta) = \frac{V^2 \cos \theta \sin \theta}{16}$$

where V is in feet per second.

Think About This Situation

Consider a golfer who consistently hits the ball so that it leaves the tee with an initial velocity of about 110 ft/sec.

a Explain why $h(t) = 0$ when $t = 0$ and when $t = \frac{V \sin \theta}{16}$.

b What is the range of the golf ball when hit at an angle of 30° to the ground?

c To get a longer hit than that in Part b, should the golfer increase or decrease the angle of the hit? Why?

d What angle of elevation do you think would produce the longest hit? Explain your reasoning.

e Describe the graph of $R(\theta)$ for a fixed initial velocity of 110 ft/sec.

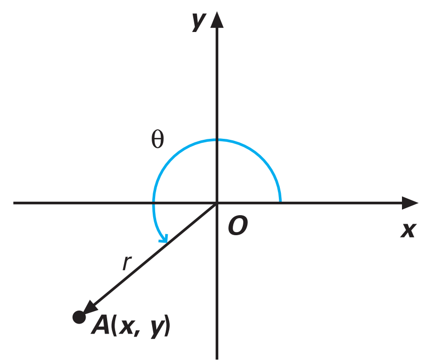
f Compare the graphs of $R(\theta)$ and $f(\theta) = \sin \theta$.

The range function $R(\theta)$ involves the product of two trigonometric functions. This product can be expressed in an equivalent form as a single trigonometric function with which you are familiar. In this lesson, you will develop key trigonometric *identities* which are useful in making such transformations.

INVESTIGATION 1 Extending the Family of Trigonometric Functions

In *Contemporary Mathematics in Context*, Course 2, the trigonometric functions, sine, cosine, and tangent, were defined as ratios of the lengths of sides of right triangles. Later the definitions for sine and cosine were extended to permit angles greater than 90° and angles less than 0°. These ideas and the definition of the tangent function are formalized below.

Consider any point $A(x, y)$, other than the origin, in a coordinate plane. Let θ be the measure of the angle determined by rotating a position vector of length r from a position along the positive x-axis to its position $\overrightarrow{OA}$. If the rotation is counterclockwise, θ is positive; if clockwise, θ is negative. The trigonometric functions are functions of θ and are defined in terms of x, y, and r where $r = \sqrt{x^2 + y^2}$, the length of $\overrightarrow{OA}$, as follows:



$$\sin \theta = \frac{y}{r} \qquad \cos \theta = \frac{x}{r} \qquad \tan \theta = \frac{y}{x},\ x \neq 0$$

For convenience, θ will be used to denote both an angle and its measure. It will be clear from the context which meaning is intended.

1. For this activity, assume that θ is an angle in **standard position** in a coordinate plane with its vertex at the origin and its *initial side* along the positive x-axis, and that $A(x, y)$ is a point on the *terminal side* of the angle.

a. Sketch θ and find $\sin \theta$, $\cos \theta$, and $\tan \theta$ for $A(x, y)$ when the coordinates of point A are as indicated.

i. (3, –2) **ii.** (–4, –3) **iii.** (2, 5) **iv.** (–3, 4)

b. Examine your results in Part a. Whether the value of a trigonometric function is positive or negative depends on the quadrant in which the terminal side of θ lies. For each quadrant in which the terminal side of an angle may lie, determine

- the corresponding range of angle measures for θ; and
- whether $\sin \theta$, $\cos \theta$, and $\tan \theta$ are positive or negative.

Summarize your findings in a chart similar to the one on the next page.

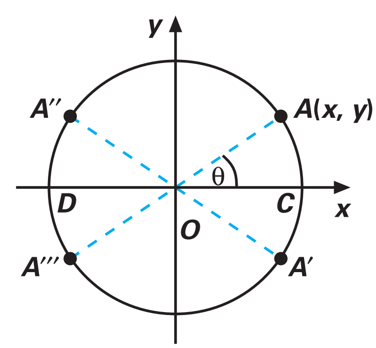
Quadrant II	Quadrant I
$90^\circ < \theta <$ ___$^\circ$ ___ $< \theta <$ ___ radians $\sin\theta > 0$ $\cos\theta$ ___ 0 $\tan\theta$ ___ 0	$0^\circ < \theta < 90^\circ$ $0 < \theta < \frac{\pi}{2}$ radians $\sin\theta > 0$ $\cos\theta > 0$ $\tan\theta > 0$
Quadrant III	**Quadrant IV**
___$^\circ < \theta <$ ___$^\circ$ ___ $< \theta <$ ___ radians $\sin\theta$ ___ 0 $\cos\theta$ ___ 0 $\tan\theta$ ___ 0	___$^\circ < \theta < 360^\circ$ ___ $< \theta < 2\pi$ radians $\sin\theta$ ___ 0 $\cos\theta$ ___ 0 $\tan\theta$ ___ 0

c. The chart in Part b does not consider the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ when the terminal side of θ is on the x- or y-axis.

- For what values of θ is its terminal side on the x-axis? What can you say about values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ in these cases?
- For what values of θ is its terminal side on the y-axis? What can you say about values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ in these cases?

2. Now suppose $A(x, y)$ is a point, other than the origin, on the terminal side of an angle θ in standard position, and $\sqrt{x^2 + y^2} = r$.

a. Find the coordinates of points A', A'', and A''' where:



- Point A' is the image of point A reflected across the x-axis.
- Point A'' is the image of point A reflected across the y-axis.
- Point A''' is the image of point A rotated 180° about the origin.

Why are points A', A'', and A''' on the circle with center O and radius r?

b. Assuming each angle is in standard position and $m\angle AOC = \theta$, express $m\angle A'OC$, $m\angle A''OC$, and $m\angle A'''OC$ in terms of θ.

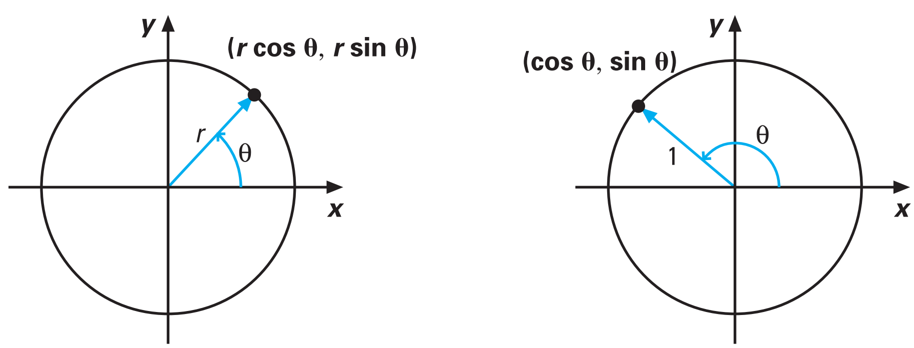
c. Investigate how knowing the value of one of the trigonometric functions of an angle θ permits you to find values of the other two trigonometric functions. Assume the terminal side of θ is in the first quadrant.

- If $\sin \theta = \frac{3}{5}$, what other angle (in a different quadrant) has the same sine? Find the cosine and tangent for both angles.
- If $\cos \theta = \frac{4}{5}$, what other angle has the same cosine? Find the sine and tangent for both angles.
- If $\tan \theta = \frac{3}{4}$, what other angle has the same tangent? Find the sine and cosine for both angles.

d. Use reasoning similar to that in Part c to complete the following:

- If $\sin \theta = -\frac{3}{5}$ and θ is in quadrant III, what other angle has the same sine? Find the cosine and tangent for both angles.
- If $\cos \theta = -\frac{4}{5}$ and θ is in quadrant II, what other angle has the same cosine? Find the sine and tangent for both angles.
- If $\tan \theta = -\frac{3}{4}$ and θ is in quadrant II, what other angle has the same tangent? Find the sine and cosine for both angles.

For any given position vector with length *r*, the terminal point of the vector is always on a circle of radius *r*. Thus, the coordinates of the terminal point are functions of the cosine and sine of the direction angle of the vector: $x = r \cos \theta$ and $y = r \sin \theta$.



In the case of a **unit circle**, a circle with radius 1 as above on the right, the coordinates are simply $x = \cos \theta$ and $y = \sin \theta$, and $\tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$, $\cos \theta \neq 0$.

3. The relationship between points on a unit circle and the values of the cosine and sine functions can be used to help sketch graphs of the trigonometric functions.

a. Sketch graphs of $y = \cos \theta$ and $y = \sin \theta$ for $-2\pi \leq \theta \leq 2\pi$ by imagining the terminal point of a position vector rotating about a unit circle. Check your graphs against calculator- or computer-produced graphs and resolve any differences.

b. Sketch a graph of $y = \tan \theta$ for $-2\pi \leq \theta \leq 2\pi$ by reasoning with the relationship $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and your graphs of the sine and cosine functions. Compare your sketch with a calculator- or computer-produced graph and resolve any differences.

c. How is the information you gathered in Activity 1 Parts b and c related to the graphs?

d. For each of the three trigonometric functions:

- state the domain and range;
- state the period.

4. You can use the graphs you produced in Activity 3 to aid in reasoning about the solutions of trigonometric equations.

a. How many values of θ satisfy the equation $\cos \theta = \frac{3}{5}$, where $0 \le \theta < 2\pi$ radians? Explain your reasoning.

b. Use the inverse cosine ($\cos^{-1}$) function on your graphing calculator to find a value of θ so that $\cos \theta = \frac{3}{5}$. Find any other values of θ between 0 and 2π satisfying this equation.

c. Solve each equation below for all values of θ, $0 \le \theta < 2\pi$.

i. $\sin \theta = \frac{1}{2}$ **ii.** $\cos \theta = -0.7$

iii. $\tan \theta = 0.5\sqrt{2}$ **iv.** $\sin \theta = 2$

d. Explain why, if a trigonometric equation involving sin θ, cos θ, or tan θ has a solution, then there is usually a second solution on the interval $0 \le \theta < 2\pi$.

e. What types of trigonometric equations might have more than two solutions on the interval $0 \le \theta < 2\pi$? Fewer than two solutions?

The reciprocals of the three trigonometric functions, sin θ, cos θ, and tan θ, give a new set of trigonometric functions called the **reciprocal functions**.

$$\text{cosecant (csc): } \frac{1}{\sin \theta} = \csc \theta$$
$$\text{secant (sec): } \frac{1}{\cos \theta} = \sec \theta$$
$$\text{cotangent (cot): } \frac{1}{\tan \theta} = \cot \theta$$

5. Explore the graph of $y = \sec \theta$ for $-2\pi \le \theta \le 2\pi$.

a. Make a sketch of the graph of $y = \cos \theta$ and of its reciprocal function $y = \sec \theta$ on the same set of axes. Identify common points.

b. Identify the values of θ, $-2\pi \le \theta \le 2\pi$, for which $y = \sec \theta$ is not defined. How is this revealed in the definition of the function? How is it seen in the graph of the function?

c. Describe the range of $y = \sec \theta$.

d. Is the function $y = \sec \theta$ periodic? If so, give the period. Why does this make sense in terms of the definition of the function?

e. Explain how you can quickly sketch the graph of $y = \sec \theta$ by remembering the shape of, and key points on, the graph of $y = \cos \theta$.

6. Repeat Activity 5 for the function $y = \sin \theta$ and its reciprocal function $y = \csc \theta$.

7. Repeat Activity 5 for the function $y = \tan \theta$ and its reciprocal function $y = \cot \theta$.

8. Investigate how reasoning with the reciprocal trigonometric functions is similar to reasoning with the trigonometric functions themselves.

a. Sketch a right triangle with sec $\theta = 2$. Evaluate the remaining five trigonometric functions of θ.

b. Sketch two position vectors such that csc $\theta = 2$. Find the remaining trigonometric function values for the direction angle of each vector.

c. Sketch two position vectors such that cot $\theta = -1$. Find the remaining trigonometric function values for the direction angle of each vector.

d. Find all solutions to each equation below on the interval $0 \leq \theta < 2\pi$. Compare the procedures you used with those used by others.

i. $\csc \theta = 4$ **ii.** $\sec \theta = -2$ **iii.** $\cot \theta = \sqrt{3}$

Checkpoint

Suppose $A(x, y)$ is a point on the terminal side of an angle θ in standard position and r is the distance from point A to the origin.

a Write the definitions of the six trigonometric functions in terms of x, y, and r.

b For what values of x and y are each of the six trigonometric functions not defined? What values of θ, $-2\pi \leq \theta \leq 2\pi$, are associated with those values of x and y? What values of θ, $-360^\circ \leq \theta \leq 360^\circ$ are associated with those values of x and y?

c Explain how sketches of the graphs of $y = \sec \theta$, $y = \csc \theta$, and $y = \cot \theta$ can be obtained from examination of the graphs of $y = \cos \theta$, $y = \sin \theta$, and $y = \tan \theta$, respectively.

Be prepared to compare your definitions of the trigonometric functions and their domains and your graphing methods to those of other groups.

On Your Own

Use symbolic reasoning to complete the following tasks.

a. Suppose $\sin \theta = -\frac{5}{13}$ and $0 \leq \theta < 2\pi$. How many solutions are there to this equation? Find all the solutions.

b. Suppose $\sin \theta = \frac{12}{13}$ and $\cos \theta = -\frac{5}{13}$. How many values of θ between 0 and 2π satisfy both equations? Find all such θ.

c. Using the definitions of the trigonometric functions recorded at the Checkpoint:

- Prove that $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cos \theta \neq 0$.
- Prove that $\cot \theta = \frac{\cos \theta}{\sin \theta}$, $\sin \theta \neq 0$.

INVESTIGATION 2 Proving Trigonometric Identities

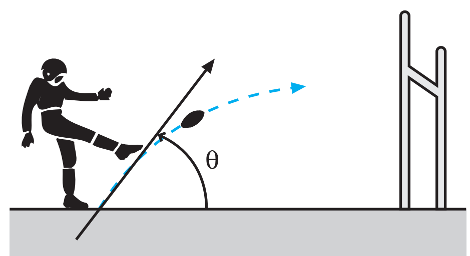
One important aspect of trigonometry is the use of many formulas expressing interrelationships among the trigonometric functions. The quotient formulas $\tan \theta = \frac{\sin \theta}{\cos \theta}$ and $\cot \theta = \frac{\cos \theta}{\sin \theta}$ are examples of *trigonometric identities*. An **identity** is a statement that is true for all replacements of a variable for which the statement is defined. You have seen other identities in your previous work in algebra. For example, $a \cdot \frac{1}{a} = 1$ is an identity for all nonzero real numbers and $\log (ab) = \log a + \log b$ is an identity for all positive real numbers. Examples of identities involving all real numbers are $a(b + c) = ab + ac$ and $(2x + 1)^2 = 4x^2 + 4x + 1$.

Identities are useful because they permit you to rewrite expressions so that they are more recognizable and often easier to interpret or work with in solving equations. Consider again the rule given at the beginning of this lesson (page 458) for the number of feet that a projectile will travel.

$$R(\theta) = \frac{V^2 \cos \theta \sin \theta}{16}$$

For a fixed value of V, R is a function of θ, the angle of the projectile.

The fact that $R(\theta)$ involves the product "$\cos \theta \sin \theta$" makes it difficult to visualize the shape of its graph or where it may have a maximum or minimum. In Modeling Task 1 (page 472), you will establish the identity $2 \cos \theta \sin \theta = \sin 2\theta$, or, equivalently, $\cos \theta \sin \theta = \frac{\sin 2\theta}{2}$. Using this identity you can rewrite $R(\theta)$ as $\frac{V^2 \sin 2\theta}{32}$, which is an easier form to interpret.



There are many additional useful identities involving trigonometric functions. You will examine several of the basic identities in this investigation, and you will learn how to prove that a statement is an identity.

1. A key identity you encountered in your previous study is the statement $(\sin \theta)^2 + (\cos \theta)^2 = 1$. For simplicity, $(\sin \theta)^2$ is often written $\sin^2 \theta$.

- **a.** Describe the graph of $f(\theta) = \sin^2 \theta + \cos^2 \theta$.
- **b.** $\sin^2 \theta + \cos^2 \theta = 1$ is called a *Pythagorean identity*. Prove this identity using one of the diagrams on page 461. Explain why the name of this identity is appropriate.
- **c.** Is $\sec^2 \theta - \tan^2 \theta = 1$ an identity?
  - If so, use the definitions of secant and tangent to prove it. If not, provide a counterexample.
  - Prove it beginning with the identity $\sin^2 \theta + \cos^2 \theta = 1$.

d. Is $\tan^2 \theta + \sec^2 \theta = 1$ an identity? If so, prove it. If not, explain how you know.

e. Is $\csc^2 \theta - \cot^2 \theta = 1$ an identity? If so, prove it. If not, explain how you know.

You now have at your disposal eight fundamental relationships that you can use to rewrite trigonometric expressions in equivalent forms.

Fundamental Relationships

Reciprocal Identities	Quotient Identities	Pythagorean Identities
$\csc \theta = \frac{1}{\sin \theta}, \sin \theta \neq 0$	$\tan \theta = \frac{\sin \theta}{\cos \theta}, \cos \theta \neq 0$	$\sin^2 \theta + \cos^2 \theta = 1$
$\sec \theta = \frac{1}{\cos \theta}, \cos \theta \neq 0$	$\cot \theta = \frac{\cos \theta}{\sin \theta}, \sin \theta \neq 0$	$1 + \tan^2 \theta = \sec^2 \theta$
$\cot \theta = \frac{1}{\tan \theta}, \tan \theta \neq 0$		$1 + \cot^2 \theta = \csc^2 \theta$

2. Suppose that in solving a problem you encounter two expressions, for example, $\sin^2 \theta$ and $(1 - \sin^2 \theta) \cdot \tan^2 \theta$, and you want to determine whether or not they are equivalent for all values θ for which both expressions are defined.

a. How could you use tables to support or contradict the claim that the equation $(1 - \sin^2 \theta) \cdot \tan^2 \theta = \sin^2 \theta$ is an identity?

b. How could you use graphs to support or contradict the claim that the equation is an identity?

c. Definitions and fundamental identities can be used to show that one expression can be transformed into the other. Justify each step in the following reasoning chain.

$$\begin{aligned}(1 - \sin^2 \theta) \cdot \tan^2 \theta &= (1 - \sin^2 \theta) \cdot \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \cos^2 \theta \cdot \frac{\sin^2 \theta}{\cos^2 \theta} \\ &= \sin^2 \theta\end{aligned}$$

d. Explain why the symbolic reasoning used above *proves* the equivalence of $\sin^2 \theta$ and $(1 - \sin^2 \theta) \cdot \tan^2 \theta$, while the approaches involving tables or graphs only make it plausible that the equation is an identity, but do not prove it.

Proving an identity involves a symbolic reasoning strategy in which each step can be justified by citing a definition or previously proved identity. The reasoning strategy used in Activity 2 Part c was to transform the left and more complicated side of $(1 - \sin^2 \theta) \cdot \tan^2 \theta = \sin^2 \theta$ into the simpler right side using known facts. Sometimes it is helpful to transform each side of a proposed identity independently into a third equivalent form, as illustrated in Activity 3.

3. Study the following proof of the identity $(1 - \sin\theta)(1 + \sin\theta) = \frac{1}{1 + \tan^2\theta}$, in which the expression on each side is transformed independently of the other. The vertical line is used to emphasize this fact.

$(1 - \sin\theta)(1 + \sin\theta)$	$\frac{1}{1 + \tan^2\theta}$
$1 - \sin^2\theta$	$\frac{1}{\sec^2\theta}$
$\cos^2\theta$	$\frac{1}{\sec\theta} \cdot \frac{1}{\sec\theta}$
	$\cos\theta \cdot \cos\theta$
	$\cos^2\theta$

Since each step on the right side can be reversed, it follows that

$$(1 - \sin\theta)(1 + \sin\theta) = \frac{1}{1 + \tan^2\theta}.$$

a. Justify each step in the independent manipulations of each side.

b. How does the fact that each step on the right side can be reversed complete the proof of the identity?

c. Why would it be inappropriate to treat the proposed identity as an equation and then use associated properties of equality to prove it?

4. Now consider the identity $\tan\theta \cdot \sin\theta = \sec\theta - \cos\theta$.

a. Prove this identity by showing each side is equivalent to $\frac{\sin^2\theta}{\cos\theta}$.

b. Prove this identity by transforming the right side into the form on the left side.

c. Look back at your two proofs. Which proof strategy do you prefer in this case and why?

5. Decide whether each of the following equations is or is not an identity. If an equation is an identity, use symbolic reasoning to prove it. If not, provide a counterexample.

a. $\sec\theta - \tan\theta\sin\theta = \cos\theta$

b. $\tan^2\theta - 2\sec\theta\sin\theta = \tan\theta(\tan\theta - 2)$

c. $\cot\theta = \cos\theta\csc\theta$

d. $\tan^2\theta = \frac{1 - \cos^2\theta}{\cos^2\theta}$

e. $\csc^2\theta - \sec^2\theta = 1$

f. $\sec^2\theta = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta}$

g. $1 - \tan\theta = \frac{\cos\theta - \sin\theta}{\cos\theta}$

h. $(1 - \sin^2\theta)(1 + \tan^2\theta) = 1$

i. $\cos(\theta_1 + \theta_2) = \cos\theta_1 + \cos\theta_2$

j. $2\sin^2\theta - 1 = 1 - 2\cos^2\theta$

k. $\sin\theta(\csc\theta - \sin\theta) = \cos^2\theta$

l. $\sec^2\theta - \csc^2\theta = \frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta}$

Checkpoint

Trigonometric identities are special types of equations involving trigonometric functions.

a How is a trigonometric identity similar to, and different from, an algebraic property like the associative property for addition?

b How is a trigonometric identity different from a trigonometric equation?

c Explain why examining tables of values and/or graphs of functions is not sufficient to prove an equation is an identity.

d Describe how you would prove an equation is an identity.

Be prepared to share your comparisons, explanations, and descriptions with the entire class.

On Your Own

Prove each statement is or is not an identity.

a. $\frac{1}{\cos\theta\sin\theta} - \frac{\cos\theta}{\sin\theta} = \tan\theta$

b. $\sin\theta = \sqrt{1 - \cos^2\theta}$

c. $\csc\theta + \cot\theta = \frac{1 + \cos\theta}{\sin\theta}$

d. $\cos^2\theta - \sin^2\theta = \cos^2\theta - \frac{1}{2}$