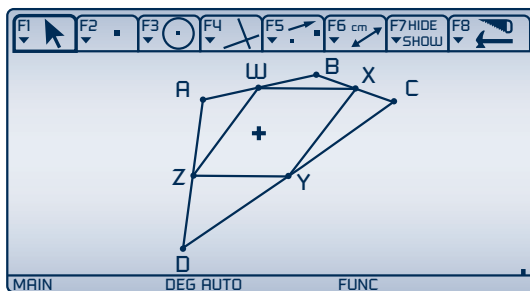


Lesson 3

Parallelograms: Necessary and Sufficient Conditions

At the beginning of this unit, you had a small set of geometric tools to use to prove the correctness of conjectures. These included the Pythagorean Theorem, the Law of Sines, and the Law of Cosines, which were proved in previous units. Now you have a large inventory at your disposal. You have properties of parallel lines and several relations among pairs of angles when two parallel lines are cut by a transversal. You have properties of similar triangles and ways to prove them similar. And you have properties of congruent triangles and methods to conclude when two triangles are congruent. These are some of the most important properties and relations in Euclidean geometry.

Properties and relations involving parallelism of lines, congruence of triangles, and similarity of triangles can help you establish properties of other polygons. In this lesson, you will explore conditions that will ensure that a quadrilateral is a parallelogram.



Think About This Situation

Using a geometry drawing program (as illustrated above) or paper and pencil, draw a quadrilateral and name it $ABCD$. Find the midpoints of the sides and connect them in order to obtain quadrilateral $WXYZ$.

- What special kind of quadrilateral does $WXYZ$ appear to be?
- Investigate this situation for special quadrilaterals such as squares, rectangles, parallelograms, trapezoids, and kites. In each case, what seems to be true about the resulting quadrilateral?
- What theorem is suggested by your observation in Parts a and b?
- What do you think the next step should be?

INVESTIGATION 1 Reasoning about Parallelograms

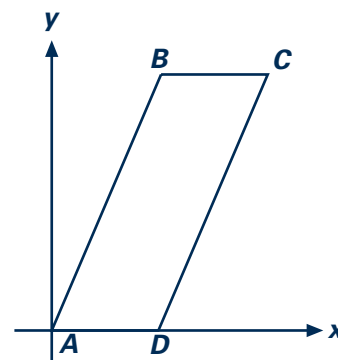
In previous courses, you have investigated applications of parallelograms and their properties. Such applications range from the design of pop-up greeting cards to the design of windshield wipers and outdoor swings. You probably have a good idea of what a parallelogram is, but in order to have a starting point for reasoning about parallelograms, everyone must agree on the same definition.



*A quadrilateral is a **parallelogram** if and only if both pairs of opposite sides are parallel.*

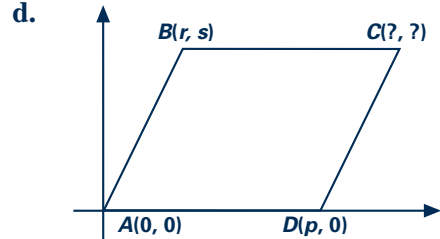
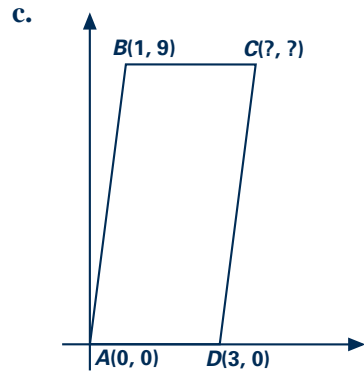
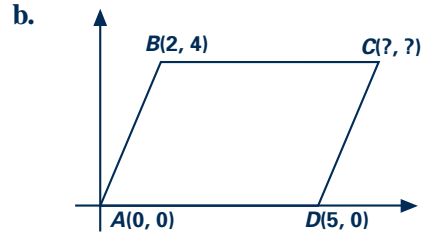
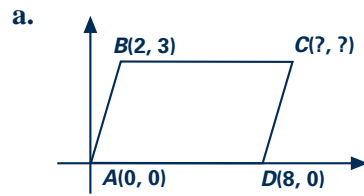
Recall that the “if and only if” in this definition allows you to use either part of the definition as given information to conclude the other part. In the following activities, you will investigate how the definition allows you to place a parallelogram on a coordinate system and specify its vertices.

1. Draw a parallelogram $ABCD$ on a coordinate system with one vertex at the origin and one side on the positive x -axis, as shown at the right.



- a. What are the coordinates of point A ? Record these on your diagram.
- b. Consider vertex D . If $AD = c$ units, what is the x -coordinate of point D ? What is the y -coordinate of point D ? Record these on your diagram.
- c. Now consider vertex B . If $AB = p$ units, is p the x -coordinate of point B ? Is the y -coordinate p ? Explain.
- d. Call the coordinates of point B (a, b) and record them on your diagram. How are sides \overline{BC} and \overline{AD} related? What is the y -coordinate of point C ? Explain your reasoning.
- e. How can you use the fact that side AB and side DC are parallel to help determine the x -coordinate of point C ? Record the coordinates of point C on your diagram.
- f. Compare your coordinate labeling of parallelogram $ABCD$ to those of other groups. Resolve any differences.

2. For each of the parallelograms shown, determine the coordinates of point C.



As you saw in Unit 3, “Symbol Sense and Algebraic Reasoning,” placing shapes on a coordinate system and assigning general coordinates to their vertices provides an algebraic tool for reasoning about the shapes.

3. Examine the following two arguments that $\overline{AB} \cong \overline{CD}$ in parallelogram $ABCD$.

Hans

First draw \overline{AC} in parallelogram $ABCD$.
 Since $\overline{AB} \parallel \overline{CD}$, $m\angle BAC = m\angle DCA$
 because they are alternate interior angles.
 Since $\overline{BC} \parallel \overline{AD}$, $m\angle BCA = m\angle DAC$.
 \overline{AC} is a corresponding side of $\triangle ABC$ and $\triangle CDA$.
 So $\triangle ABC \cong \triangle CDA$ by ASA .
 It follows by corresponding parts that $\overline{AB} \cong \overline{CD}$.

Kara

First place $ABCD$ on a coordinate system with point A at the origin and point D on the x -axis.
 The vertices have coordinates $A(0, 0)$, $B(a, b)$, $C(a + c, b)$, and $D(c, 0)$.
 Using the distance formula:

$$AB = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$$

$$CD = \sqrt{(a + c - c)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$$
 Thus, $AB = CD$ or $\overline{AB} \cong \overline{CD}$.

- a. Is Hans' argument valid? What previously proved information did he use?
- b. Is Kara's argument valid? What information did she use to make her argument?
- c. Could Hans conclude other facts about parallelograms once he deduced $\triangle ABC \cong \triangle CDA$? What other facts?
- d. Could Kara also conclude other facts about parallelograms once she used the distance formula to get $AB = CD$? What other facts?
- e. Whose proof is easiest to follow and why? Which of these methods of proof would you have used and why?

Hans' proof is called a **synthetic proof** because it applies to a figure with no coordinates. Kara's argument is called an **analytic** or **coordinate proof** since it uses coordinates. Sometimes one argument form is easier and sheds more light on other possible deductions than the other.

4. Listed below are conjectures about properties of every parallelogram. Examine each to see if your group thinks it is correct. If you think it is correct, construct a proof. If you think it is incorrect, give a counterexample. Share the work among your group members. Then discuss each other's proofs.

If a quadrilateral is a parallelogram, then the following is true:

- a. Opposite angles are congruent.
- b. Opposite sides are congruent.
- c. Adjacent angles are supplementary.
- d. All angles are congruent.
- e. Each diagonal divides the shape into two congruent triangles.
- f. The diagonals bisect each other.
- g. The diagonals are congruent.
- h. The diagonals are perpendicular.



Checkpoint

In addition to its recognizable shape, a parallelogram has many geometric properties that can be proved.

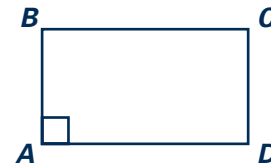
- a. Make a list summarizing the properties about the sides, angles, and diagonals of a parallelogram that you were able to prove.
- b. How do synthetic and analytic arguments differ?

Be prepared to share your list and explanation with the entire class.

On Your Own

A **rectangle** may be defined as a parallelogram with a right angle.

Since a rectangle is a parallelogram, everything about parallelograms is also true about rectangles. Using inductive reasoning, make a conjecture about a property of all rectangles that is not also true for all parallelograms. Develop a proof for your conjecture.



You now have a list of theorems that give properties of parallelograms. These are useful whenever you know that a quadrilateral is a parallelogram. If a quadrilateral is *not* known to be a parallelogram, then you cannot conclude that any of those properties hold. In the remainder of this investigation, you will explore conditions that ensure that a quadrilateral is a parallelogram. You already know one condition from the definition of a parallelogram. Namely, if a quadrilateral has two pairs of opposite sides parallel, then the quadrilateral is a parallelogram.

5. Examine this conjecture that Shimo made after drawing and analyzing a few quadrilaterals.

If a quadrilateral has two pairs of opposite sides congruent, then the quadrilateral is a parallelogram.

- Experiment with drawings to see if this conjecture makes sense.
 - Construct a proof or a counterexample of Shimo's conjecture.
6. Here are additional conditions that may or may not ensure that a quadrilateral is a parallelogram. Investigate each conjecture by drawing shapes satisfying the conditions and checking to see if the quadrilateral is a parallelogram. Share the workload among members of your group. Compile a group list of conditions that seem to guarantee that a quadrilateral is a parallelogram.
- If a quadrilateral has two pairs of opposite angles congruent, then it is a parallelogram.
 - If a quadrilateral has one pair of opposite angles and a pair of opposite sides congruent, then it is a parallelogram.
 - If a quadrilateral has diagonals that bisect each other, it is a parallelogram.
 - If a quadrilateral has one pair of opposite sides parallel and the other pair of opposite sides congruent, then it is a parallelogram.
 - If a quadrilateral has two distinct pairs of adjacent sides congruent, then it is a parallelogram.
 - If a quadrilateral has one pair of opposite sides congruent and parallel, then it is a parallelogram.
7. For each condition you listed in Activity 6, construct a proof that your conjecture is correct. Share the workload and then discuss the proofs within your group.

Checkpoint

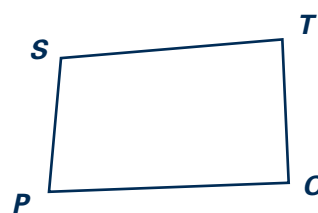
Make a list of several conditions on a quadrilateral that guarantee it to be a parallelogram.

Be prepared to demonstrate to the class a proof of each condition in your list.

On Your Own

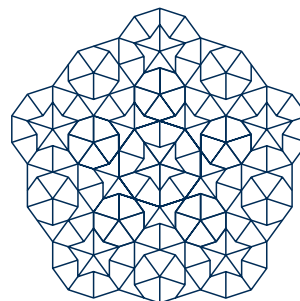
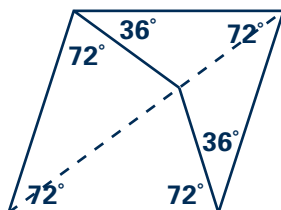
The figure $STOP$ is a quadrilateral. For each set of conditions, explain why the conditions do or do not ensure that $STOP$ is a parallelogram.

- $\angle S \cong \angle O$ and $\angle T \cong \angle P$
- $\overline{ST} \parallel \overline{OP}$ and $\overline{TO} \cong \overline{SP}$
- $\overline{ST} \cong \overline{OP}$ and $\overline{TO} \cong \overline{SP}$
- $\angle S \cong \angle O$ and $\overline{ST} \cong \overline{OP}$
- $\overline{ST} \cong \overline{TO}$ and $\overline{OP} \cong \overline{PS}$
- M is the midpoint of \overline{SO} and of \overline{TP}



INVESTIGATION 2 Special Kinds of Parallelograms

One special parallelogram, with a pair of congruent adjacent sides, is a rhombus. Since every parallelogram has its *opposite* sides congruent, it follows that a rhombus has four congruent sides. (Why?) The rhombus shown at the left below can be used to make a pair of tiles called a *dart* and a *kite*. In the first course of *Contemporary Mathematics in Context*, you may have discovered that a rhombus will form a tiling of the plane that has translation symmetry. What is interesting about these particular figures, discovered by Roger Penrose, a professor at the University of Oxford, is that copies of these two shapes can be used to form a tiling, a portion of which is shown at the right below, which does *not* have translation symmetry.





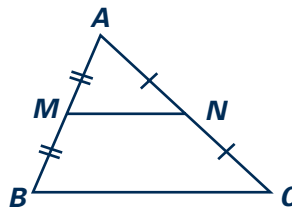
Penrose Tiles quilt made by Elaine Krajenke Ellison

Penrose's discovery is based on a particular rhombus. In the next two activities, you will explore additional properties of any rhombus.

1. Sketch several different rhombuses with and without their diagonals.
 - a. What are some properties of a rhombus that are consequences of the fact that a rhombus is a parallelogram?
 - b. Now visually inspect your sketches. Make a group list of other characteristics observed in the sketches that might be correct for all rhombuses.
 - c. Write each of your conjectures in if-then form.
 - d. Construct proofs for your conjectures. Share the work and then discuss the proofs within your group.
2. Given below are various conditions on a parallelogram. Determine if each condition is sufficient to conclude that the parallelogram is a rhombus. For each condition that seems to ensure that the parallelogram is a rhombus, write the corresponding if-then statement and a proof.
 - a. The diagonals bisect each other.
 - b. The diagonals are perpendicular.
 - c. The diagonals bisect the angles.
 - d. A pair of adjacent angles are congruent.

Now refer back to the "Think About This Situation" at the beginning of this lesson, page 325. You and your classmates may have conjectured that the shape formed by connecting the midpoints of consecutive sides of a quadrilateral is a parallelogram. In the remainder of this investigation, you will examine this situation further.

3. Consider first the case of connecting midpoints of two sides of a triangle.
 - a. Draw several triangles and locate the midpoints of two sides for each one. Connect the midpoints in each. What appears to be true about the segment determined by the midpoints?
 - b. In $\triangle ABC$, M is the midpoint of \overline{AB} and N is the midpoint of \overline{AC} . How does $\triangle AMN$ appear to be related to $\triangle ABC$? Prove that your observation is correct.



- c. Given your deduction in Part b, prove that $\vec{MN} \parallel \vec{BC}$.
- d. What can you conclude about the relative lengths of \vec{MN} and \vec{BC} ? Why?

The theorem you proved in Activity 3 Parts c and d is often called the **Midpoint Connector Theorem for Triangles**.

4. The midpoint connector theorem which you conjectured for quadrilaterals may be stated as follows:

If the midpoints of consecutive sides of any quadrilateral are connected, the resulting quadrilateral is a parallelogram.

- Draw a diagram representing the **Midpoint Connector Theorem for Quadrilaterals**.
- Write a proof of this theorem.

Checkpoint

As quadrilaterals become more specialized, they exhibit properties in addition to the properties for more general cases.

- List two properties of every rhombus that are not properties of every parallelogram.
- How does your work in this lesson with the Midpoint Connector Theorem for Quadrilaterals illustrate how inductive reasoning and deductive reasoning support each other?
- A rectangle is a special kind of quadrilateral. If the midpoints of consecutive sides of any rectangle are connected, is a special kind of parallelogram formed? Write an argument justifying your conclusion.

Be prepared to share your responses and reasoning with the entire class.

On Your Own

Consider the following information about quadrilateral $ABCD$.

- $\overline{AC} \perp \overline{BD}$
- \overline{AC} bisects $\angle A$ and $\angle C$

- Make several sketches of quadrilaterals satisfying these conditions.
- Do the sketches suggest $ABCD$ is a parallelogram? A rhombus?
- State one conjecture about $ABCD$ that you think is always true.
- Prove your conjecture.