

Lesson 4

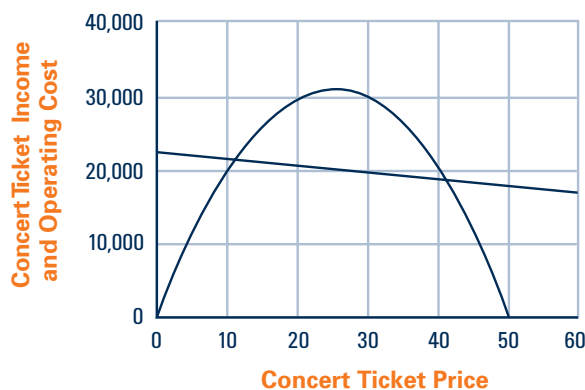
Reasoning to Solve Equations and Inequalities

In earlier work in this unit, you modeled situations with several variables and equations. For example, suppose you were given business plans for a concert showing how operating cost and ticket sale income were expected to relate to ticket price x .

$$\text{Cost: } C(x) = 22,500 - 100x$$

$$\text{Income: } I(x) = 2,500x - 50x^2$$

A graph of those two functions looks like this:



Think About This Situation

Questions important to planning the concert can be answered by solving equations and inequalities involving the cost and income functions.

- What would you learn from solutions of the following?
 - $I(x) = C(x)$
 - $I(x) > C(x)$
 - $I(x) < C(x)$
- How would you solve the equation and inequalities in Part a using the following tools?
 - Technology
 - Reasoning about the symbolic function rules (without the help of calculator tables or graphs)

INVESTIGATION 1 Reasoning about Linear Equations and Inequalities



Situations involving comparison of business plans often involve linear functions, leading to questions requiring the solution of linear equations and inequalities. For example, a pizza company considering lease options for a delivery truck might have choices like those shown below. In each case, the lease cost (in dollars) is a function of lease time (in weeks).

$$\text{Plan A: } A(t) = 2,500 + 75t$$

$$\text{Plan B: } B(t) = 1,000 + 90t$$

1. What do the numbers 2,500 and 1,000 tell about the conditions of each lease? What do the numbers 75 and 90 tell?
2. At the start, Plan A is more expensive than Plan B. To see when the lease costs might be equal, you could solve the equation $2,500 + 75t = 1,000 + 90t$. One way to solve that equation is to reason like this:

$$2,500 + 75t = 1,000 + 90t$$

$$1,500 + 75t = 90t$$

$$1,500 = 15t$$

$$100 = t$$

- a. Justify each step in the solution process.
 - b. How can you check that $t = 100$ is the solution? What does $t = 100$ tell about the truck-lease plans?
 - c. How could you arrive at the same result with a different sequence of steps?
3. Now, consider the inequality $2,500 + 75t < 1,000 + 90t$.
 - a. What will a solution to this inequality tell about the truck-leasing situation?
 - b. Why does each step in the following reasoning make sense?

$$2,500 + 75t < 1,000 + 90t$$

$$2,500 < 1,000 + 15t$$

$$1,500 < 15t$$

$$100 < t$$

- c. What does the solution $100 < t$ tell about the truck-leasing situation?
- d. How can you check the solution?
- e. Use similar reasoning to solve the inequality $2,500 + 75t > 1,000 + 90t$, and explain what the solution tells about the truck-leasing situation.

In solving the equations and inequalities that compare two truck-lease plans, it is helpful to keep in mind what the parts of each expression mean. Now try to use similar reasoning patterns to solve equations and inequalities without such clues.

4. Solve each of the following equations by symbolic reasoning alone. Record each step in your reasoning. Check your solutions.

a. $2x + 15 = 45 + x$

b. $10 - 4x = 3x - 4$

c. $7x - 11 = 10$

d. $-6x - 15 = 4x + 10$

e. $25 = 10 - 5x$

5. Solve each of the following inequalities by symbolic reasoning alone. Record each step in your reasoning. Check your answers.

a. $3x + 10 < 14$

b. $13 + 5x > 23$

c. $8x + 12 < 46$

d. $80 + 6x > 200$

6. Here are three solutions of inequalities that lead to incorrect results. In each case, show with function tables or graphs that the proposed solutions are not correct. Then find and correct the error in the reasoning process.

a. Solve: $5x + 20 < 3x$

$$20 < -2x$$

$$-10 < x$$

b. Solve: $11x - 19 < 15x + 17$

$$11x < 15x + 36$$

$$-4x < 36$$

$$x < -9$$

c. Solve: $10 + 9x < 3x - 8$

$$18 + 9x < 3x$$

$$18 < -6x$$

$$-3 < x$$

7. Solve each of the following inequalities by symbolic reasoning alone. Show each step in your reasoning. Check your answers.

a. $3x + 10 < 5x + 4$

b. $23 - 5x > 7x - 13$

c. $8x + 12 < 46 - 9x$

d. $80 + 6x > 21x - 15$

Two equations or inequalities are called **equivalent** if they have identical solutions. One strategy for solving linear equations and inequalities is to start with the given equation or inequality and construct a sequence of simpler forms, each equivalent to its predecessor, until you get an equation or inequality so simple that the solution is obvious. The challenge is to find ways of writing equivalent equations and inequalities that do become progressively simpler.

8. Which of the following pairs of equations and inequalities are equivalent? Explain your reasoning in each case.

- a. $3x + 2 = 5$ and $3x = 3$
- b. $7x - 8 = 12 + 3x$ and $4x = 20$
- c. $\frac{1}{3}x + 9 = 6$ and $x + 9 = 18$
- d. $10x + 15 = 35$ and $2x + 3 = 7$
- e. $10x + 15 = 35$ and $10x = 20$
- f. $3x + 2 < 5$ and $3x < 3$
- g. $7x - 8 > 12 + 3x$ and $4x > 20$
- h. $10x + 15 < 35$ and $2x + 3 < 7$
- i. $10x + 15 > 35$ and $10x > 20$

9. Look back over the pairs of equations and inequalities in Activity 8 and your answers to the equivalence question. What operations on equations and inequalities seem likely to produce simpler equivalent forms?

Checkpoint

Many situations call for comparing two linear functions like the following:

$$f(x) = a + bx \quad g(x) = c + dx$$

- a** What overall strategy and specific reasoning steps would you use to solve an equation of the form $a + bx = c + dx$? Explain how you could check the solution.
- b** What overall strategy and specific reasoning steps would you use to solve an inequality of the form $a + bx < c + dx$? How could you check the answer?
- c** How do the graphs of expressions like $y = a + bx$ and $y = c + dx$ illustrate solutions to the equations and inequalities described in Parts a and b? How would those solutions appear in tables of values for the two functions?

Be prepared to explain your strategies and reasoning to the entire class.



On Your Own

Two cellular telephone service plans offer monthly costs (in dollars) that are functions of time used (in minutes) with the following rules:

$$B(t) = 35 + 0.30t$$

$$C(t) = 25 + 0.50t$$

Write and solve (without use of technology) equations and inequalities that help in answering these questions:

- Under what conditions will Plan *B* cost less than Plan *C*?
- Under what conditions will Plan *B* cost the same as Plan *C*?
- Under what conditions will Plan *C* cost less than Plan *B*?

In each case, show how you can use a calculator to check your solutions.

INVESTIGATION 2 Reasoning about Quadratic Equations and Inequalities

Two key questions are often associated with quadratic function models of the form $f(x) = ax^2 + bx + c$:

- What is the maximum (minimum) value and where does it occur?
- For what values of the input variable x will $f(x) = 0$?

In the case of the concert-planning model described at the start of this lesson, the quadratic income function was $I(x) = 2,500x - 50x^2$. The two key questions can be stated in this way:

- What ticket price will lead to maximum projected income from ticket sales?
- What ticket prices will produce no projected ticket income at all?

You can answer both questions by scanning a table or graph of $(x, I(x))$ values. But you can also get the answers easily by using algebraic reasoning.

- Justify each step in the following analysis of the concert-income situation.
 - Solving the equation $2,500x - 50x^2 = 0$ will help.
 - The equation in Part a is equivalent to $50x(50 - x) = 0$.
 - $50x(50 - x) = 0$ when $x = 0$ or when $x = 50$.
 - The maximum income will occur when $x = 25$.
 - That income is \$31,250.

2. Unfortunately, many quadratic equations are not easy to solve using the type of factoring that worked so well in Activity 1. For example, consider the problem of finding projected break-even prices for the planned concert. Those are the prices for which income from ticket sales will equal expenses for operating costs. Since the cost equation for this situation was $C(x) = 22,500 - 100x$, this problem requires solving the equation $2,500x - 50x^2 = 22,500 - 100x$.

a. You can start by writing an equivalent equation with a quadratic expression equal to 0:

$$-50x^2 + 2,600x - 22,500 = 0$$

Why is this equation equivalent to the original?

b. Factor the left side of this equation to get $-50(x^2 - 52x + 450) = 0$. Why is this factored form equivalent to the equation in Part a?

The form of the equation given in Part b does not look easy to continue to solve by factoring!

There is another way you can solve quadratic equations, even when factoring seems impossible. You can use the **quadratic formula**. If $ax^2 + bx + c = 0$ (and $a \neq 0$), then the **roots** of the equation are

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

or, writing these separately,

$$x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

(You'll explore a derivation of this formula in Lesson 5.)

3. Solve the break-even equation $-50x^2 + 2,600x - 22,500 = 0$ using the quadratic formula.

a. Give the values for a , b , and c .

b. Evaluate $-\frac{b}{2a}$.

c. Evaluate $\frac{\sqrt{b^2 - 4ac}}{2a}$.

d. Now calculate $x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$.

e. Describe at least three different ways to check your calculated roots in Part d. Check the roots in the original equation using one of those methods.

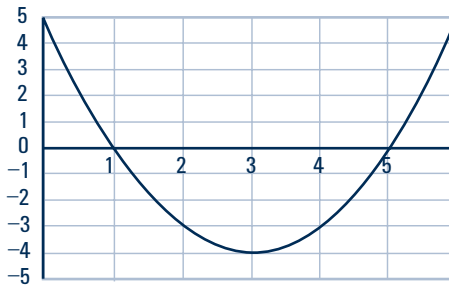
f. Use the quadratic formula to solve the equation $x^2 - 52x + 450 = 0$. Compare the result to the answer in Part d and explain similarities or differences.

4. Some computer software and some calculators have a “solve” feature that allows you to solve equations directly, if one side of the equation is equal to 0. The procedure for using these solving capabilities varies. You may need to consult your manual to learn how to use the feature.

EQUATION SOLVER
 eqn : $0 = -50x^2 + 2600$
 $x = 22500$

Use the solve feature on your calculator or computer software to check your solutions to Parts d and f of Activity 3.

5. Now consider the quadratic equation $x^2 - 6x + 5 = 0$. A graph of the function $f(x) = x^2 - 6x + 5$ is shown below.



- Give the values for a , b , and c that should be used to solve $x^2 - 6x + 5 = 0$ with the quadratic formula.
 - Evaluate $-\frac{b}{2a}$.
 - Evaluate $\frac{\sqrt{b^2 - 4ac}}{2a}$.
 - Now calculate $x = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}$ and $x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$.
 - Compare the quadratic formula calculations to the graph of $f(x) = x^2 - 6x + 5$. What information is provided by the expression $-\frac{b}{2a}$? By the expression $\frac{\sqrt{b^2 - 4ac}}{2a}$?
6. Use the quadratic formula to solve each of the following quadratic equations. Then try to solve the same equations by factoring. In each case, check your work by using the solve feature on your calculator or computer software or by substituting your proposed roots for x into the equation.
- $-2x^2 - 3x + 7 = 0$
 - $5x^2 - x - 4 = 0$
 - $3x^2 - 2x + 1 = 0$
 - $-x^2 + 2x - 3 = 0$
 - $4x^2 + 12x + 9 = 0$

7. Now look back at your work in Activity 6 and search for connections between the quadratic formula calculations and the graphs of the corresponding function rules.

- Explain the special significance of the equation $x = -\frac{b}{2a}$ for a quadratic function with rule in the form $f(x) = ax^2 + bx + c$.
- What information is provided by the expression $\frac{\sqrt{b^2 - 4ac}}{2a}$?

Test your ideas in each of the following cases by graphing the function and the vertical line $x = -\frac{b}{2a}$. (If you choose to use your calculator rather than sketching your graph, consult your manual as needed.)

a. $f(x) = 2x^2 + 4x - 9$

b. $f(x) = 3x^2 - 2x - 5$

c. $f(x) = x^2 + 6x - 10$

d. $f(x) = -x^2 + 2x - 9$

8. Think about the ways in which the graph of $f(x) = ax^2 + bx + c$ could intersect the x -axis. How many possible roots could the equation $ax^2 + bx + c = 0$ have?

a. Use the quadratic formula to solve each equation and identify the step that first shows the number of roots you can expect.

■ $x^2 + 8x + 12 = 0$

■ $x^2 + 8x + 16 = 0$

■ $x^2 + 8x + 20 = 0$

b. Sketch graphs of the quadratic functions corresponding to the three equations above, and explain how those graphs show the number of roots in each case.

9. Suppose that $f(x)$, $g(x)$, $j(x)$, and $h(x)$ are quadratic functions with the zeroes indicated below. Find values of x for which each of these functions would have maximum or minimum values. Then write possible rules for the functions in factored form.

a. $f(6) = 0$ and $f(-2) = 0$

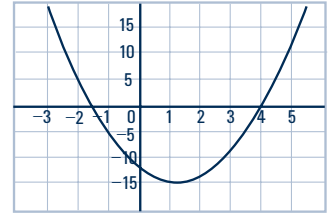
b. $g(-7) = 0$ and $g(3) = 0$

c. $j(-2) = 0$ and $j(-5) = 0$

d. $h(2) = 0$ and $h(4.5) = 0$

10. Explain how the quadratic formula can help you determine the minimum value of the function $f(x) = 4x^2 - 7x - 10$.

11. The quadratic formula and other equation-solving methods can be used to solve quadratic inequalities as well. Examine the graph of the function $f(x) = 2x^2 - 5x - 12$ at the right. It has zeroes at $x = -1.5$ and $x = 4$.



- a. Explain how the number line graph below shows the solution of the quadratic inequality $2x^2 - 5x - 12 < 0$.



- b. How does the number line graph in Part a relate to the graph of the function $f(x) = 2x^2 - 5x - 12$?
- c. Make a number line graph showing the solution of the quadratic inequality $2x^2 - 5x - 12 > 0$.
- d. How would you modify the number line graph in Part c to show the solution of the inequality $2x^2 - 5x - 12 \geq 0$?
12. Using Activity 11 as an example, if needed, solve the following quadratic inequalities.
- | | |
|---------------------------|-------------------------|
| a. $x^2 - x - 6 < 0$ | b. $x^2 - x - 6 > 0$ |
| c. $-x^2 + 5x + 6 > 0$ | d. $-2x^2 - 3x + 5 < 0$ |
| e. $x^2 - 8x + 16 \leq 0$ | f. $x^2 - 8x + 16 > 0$ |

Checkpoint

Quadratic functions, with graphs that are parabolas, can be written with symbolic rules in the form $f(x) = ax^2 + bx + c$. You have learned to solve the related quadratic equations $ax^2 + bx + c = 0$ using the quadratic formula.

- Explain the steps that you would take to determine the zeroes and the minimum value of the function $f(x) = 3x^2 - 2x - 8$.
- What are the advantages and disadvantages of solving quadratic equations by factoring? By using the quadratic formula? By using the solve feature of your calculator or computer software?
- How can you use the quadratic formula or the solve feature to find a factored form of a quadratic function rule?
- How does use of the quadratic formula show whether a given equation will have 2, 1, or 0 roots? How will this information appear in a graph?

Be prepared to share your methods and thinking with the class.

 **On Your Own**

Use what you have learned about the quadratic formula to complete the following tasks.

a. Find the zeroes and the lines of symmetry for the graphs of the following functions.

■ $f(x) = x^2 - 4x + 1$

■ $g(x) = x^2 + 6x - 11$

■ $h(x) = x^2 - 24$

b. For each of the following functions, find the minimum or maximum value of the function.

■ $f(x) = x^2 - 3x + 9$

■ $g(x) = -x^2 + 8x + 2$

■ $h(x) = x^2 - 49$

c. Graph solutions for these quadratic inequalities:

■ $x^2 - 3x - 4 > 0$

■ $x^2 + x - 6 < 0$

■ $-x^2 - 2x + 3 > 0$