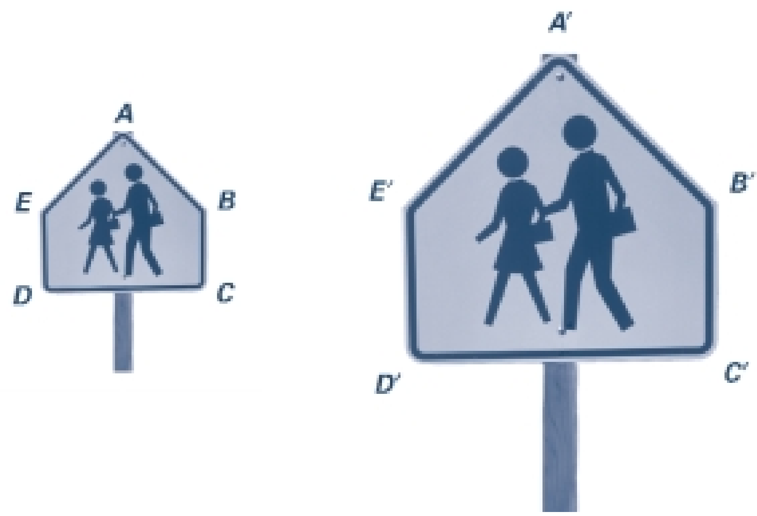
INVESTIGATION 2 What's the Angle?

In the previous investigation, you learned that when the rigidity property of triangles is combined with the ability to adjust the length of a side, the opportunities for useful application expand greatly. You probably noticed that the methods used to determine lengths and angle measures involved measuring the models you made. In this investigation, you will use right triangles and similarity to explore other ways in which lengths and angle measures can be determined.

Recall that if two figures are *similar* with a scale factor of *s*, then corresponding angles are congruent (≅) and lengths of the corresponding sides are related by the multiplier *s*. The two school crossing signs shown below are similar.



Here pentagon *ABCDE is similar to A′B′C′D′E′* ($ABCDE \sim A'B'C'D'E'$) with a scale factor of 2. $\angle B$ corresponds to $\angle B'$, so $\angle B \cong \angle B'$. (*$\angle B$ is congruent to $\angle B'$*.) Segment ED corresponds to segment $E'D'$, so $2 \cdot ED = E'D'$ or, equivalently, $ED = \frac{E'D'}{2}$.

1. Imagine that you and a classmate each draw a triangle with three angles of one triangle congruent to three angles of the other triangle. Do you think the two triangles will be similar? Make a conjecture.

a. Now conduct the following experiment. Have each member of your group draw a segment (no two with the same length). Use a protractor to draw a 50° angle at one end of the segment. Then draw a 60° angle at the other end of the segment to form a triangle.

- What should be the measure of the third angle? Check your answer.
- Are these triangles similar to one another? What evidence can you give to support your view?

b. Repeat Part a with angles measuring 40° and 120°. Are these triangles similar? Give evidence to support your claim.

c. Which, if any, of the following statements do you think are always true? Justify your response with reasons or a counterexample.

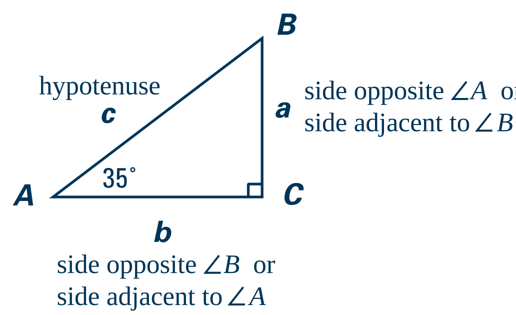
- If one triangle has three angles congruent to three corresponding angles on another triangle, then the triangles are similar.
- If one triangle has two angles congruent to two corresponding angles on another triangle, then the triangles are similar.
- If one triangle has one angle congruent to one angle on another triangle, then the triangles are similar.

2. Now apply your discoveries in Activity 1 to the special case of right triangles.

a. Each group member should draw a segment AC (each a different length). Using your segment AC as a side, draw $\triangle ABC$ with $\angle A$ measuring 35˚ and a right angle at C. It is important to draw your triangle very carefully. What is the measure of the other angle ($\angle B$)?

b. Are the triangles your group members drew similar? Explain.

c. Choose the smallest triangle drawn in Part a. Determine the approximate scale factors relating this triangle to the others drawn by group members.



For a right triangle ABC, it is standard procedure to label the right angle with the capital letter C and to label the sides opposite the three angles lower case a, b, and c as shown. Complete a labeling of your triangle in this way. (Additional ways to refer to the sides of a right triangle are also included in the diagram. The **hypotenuse** is always the side opposite the right angle, but the designation of the other sides depends on which angle is considered.)

3. A diagram of a right $\triangle ABC$ is given below. Give the measures of the following angles or sides:

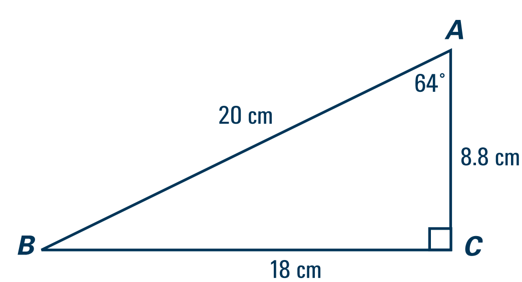
a. $\angle C$

b. $\angle B$

c. Side opposite $\angle A$

d. Side (leg) adjacent to $\angle B$

e. Side (leg) adjacent to $\angle A$



4. Refer to the right triangles your group drew for Activity 2.

   a. Make a table like the one below. Each group member should choose a unit of measure. Then carefully measure and calculate the indicated ratios for the right $\triangle ABC$ that you drew. Express the ratios to the nearest 0.01. Investigate patterns in the three ratios for your group's triangles.

Right Triangle Side Ratios

Ratio	Student 1	Student 2	Student 3	Student 4
$\frac{a}{c}$	________	________	________	________
$\frac{b}{c}$	________	________	________	________
$\frac{a}{b}$	________	________	________	________

   b. Compare the ratios from your group with those of other groups.

   c. On the basis of Parts a and b, make a conjecture about the three ratios in the table for any right $\triangle ABC$ with a 35° angle at A.

   d. How could the three ratios be described in terms of the hypotenuse and the sides opposite and adjacent to $\angle A$? In terms of the hypotenuse and the sides opposite and adjacent to $\angle B$?

   e. Make a conjecture about the three ratios in the table for any right triangle with a 35° angle.

   f. Make a conjecture about the three ratios for any right triangle with a 55° angle.

5. As a group, draw several examples of a right $\triangle ABC$ in which $\angle A$ has a measure of 40° and $\angle C$ has a measure of 90°.

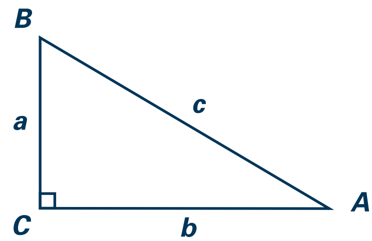
   a. Compute the three ratios $\frac{a}{c}$, $\frac{b}{c}$, and $\frac{a}{b}$. Record the ratios in a table like the one in Activity 4.

   b. What pattern do you see in these ratios?

   c. How is the pattern for these ratios similar to the pattern for the ratios in Activity 4? How is it different?

   d. What seems to cause the differences in the results from the two activities? Test your conjecture by experimenting with another set of similar right triangles and inspecting the ratios.

You have observed that as the measure of an acute angle changes in a right triangle, ratios of the lengths of the sides also change. In fact, each ratio is a function of the size of the angle. These relationships are important because they relate measures of angles (in degrees) to ratios of linear measures (in centimeters, miles, and so on). The relationships or functions have special names. For a right triangle ABC with sides a, b, and c, the **sine**, **cosine**, and **tangent** ratios for $\angle A$ are defined as follows:



$$\text{sine of } \angle A = \sin A = \frac{\text{length of side opposite } \angle A}{\text{length of hypotenuse}} = \frac{a}{c}$$

$$\text{cosine of } \angle A = \cos A = \frac{\text{length of side adjacent to } \angle A}{\text{length of hypotenuse}} = \frac{b}{c}$$

$$\text{tangent of } \angle A = \tan A = \frac{\text{length of side opposite } \angle A}{\text{length of side adjacent to } \angle A} = \frac{a}{b}$$

These ratios are called **trigonometric ratios**.

Sine B, cosine B, and tangent B are similarly defined by forming the ratios using the sides opposite and adjacent to $\angle B$. The abbreviations are $\sin B$, $\cos B$, and $\tan B$.

6. Refer to the triangles you drew for Activity 5. Write the definitions for $\sin B$, $\cos B$, and $\tan B$, and then find $\sin 50°$, $\cos 50°$, and $\tan 50°$.

7. Suppose you have a right triangle with an acute angle of 27.5°. One way you could find the sine, cosine, and tangent of 27.5° would be to make a very accurate right triangle and measure. Another way is to use your calculator.

a. To calculate a trigonometric ratio for an angle measured in degrees, first be sure your calculator is set in *degree* mode. Then simply press the keys corresponding to the ratio desired. For example, to calculate $\sin 27.5°$ on most graphing calculators, press SIN 27.5 ENTER. Try it. Then calculate $\cos 27.5°$ and $\tan 27.5°$.

b. Compare $\sin 27.4°$ and $\sin 27.6°$ to your value for $\sin 27.5°$. How many decimal places should you include to show that the angle whose sine you are finding was measured to the nearest 0.1°?

c. How many decimal places should you report for $\sin 66.5°$ to indicate that the angle was measured to the nearest 0.1°?

d. Use your calculator to find the sine, cosine, and tangent of 35° and of 50°. Compare these results with those you obtained by measuring in Activities 4 through 6.

You can use your calculator to compute values for sine, cosine, and tangent of any angle. Several hundred years ago mathematicians spent years calculating these ratios by hand to several decimal places so that they could be used in surveying and astronomy. Until recently, before scientific and graphing calculators became common, people usually looked up the ratio values from a large table. Now that a calculator replaces this tedious work, you can concentrate on understanding trigonometric ratios and their uses.

Checkpoint

Knowing some information about a triangle or pair of triangles often allows you to conclude other information.

a What can you conclude about two triangles if two angles of one are congruent to two angles of the other?

b Why does knowing the measure of an acute angle of a right triangle completely determine the triangle's shape?

c For two different triangles *ABC* and *DEF*, in which $\angle A$ and $\angle D$ both have measure $x°$ and $\angle C$ and $\angle F$ are right angles, what can you say about the ratios $\frac{AC}{AB}$ and $\frac{DF}{DE}$? Explain why this makes sense.

d If $\sin x° = \frac{b}{c}$ in right $\triangle ABC$, then which angle has measure $x°$? Which angle has measure $x°$ when $\cos x° = \frac{b}{c}$?

Be prepared to discuss your responses with the entire class.

On Your Own

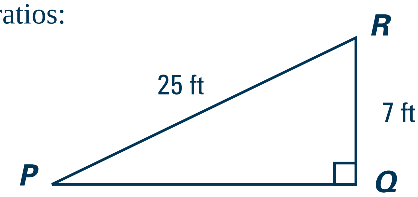
Refer to the drawing of $\triangle PQR$ below.

a. Which is the side opposite $\angle P$? The side adjacent to $\angle R$? The hypotenuse?

b. Use the Pythagorean Theorem to find the length of side *PQ*.

c. Find the following trigonometric ratios:

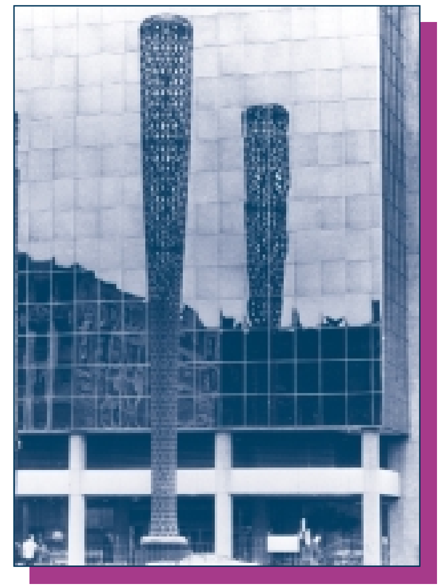
- $\sin P$
- $\sin R$
- $\tan P$
- $\cos R$



d. Valerie measured $\angle R$ and found it to be almost 74°. Use a calculator and your results from Part c to estimate the measure of $\angle R$ to the nearest 0.1°.

INVESTIGATION 3 Measuring Without Measuring

Shown below is Chicago's *Bat Column*, a sculpture by Claes Oldenburg.



1. In your group, brainstorm about possible ways to determine the height of the sculpture.

a. Choose one method and write a detailed plan.

b. Trade plans with another group and compare the two plans.

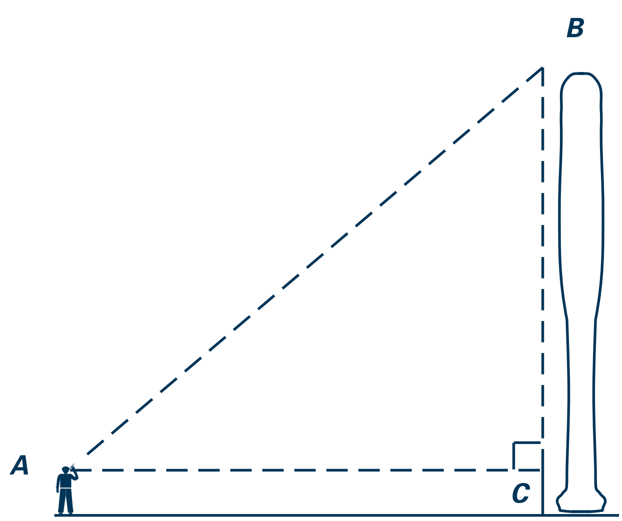
c. What assumptions did the other group make in devising its plan?

d. Which plan seems easier to carry out? Why?

Your class probably thought of several plans to determine the height of *Bat Column*. For example, one could use an extension ladder on a fire truck to climb to the top and drop a weighted and measured cord to the ground. This would be a *direct measurement* procedure.

2. An *indirect* way to measure the height of *Bat Column* would be to use a right triangle and a trigonometric ratio.

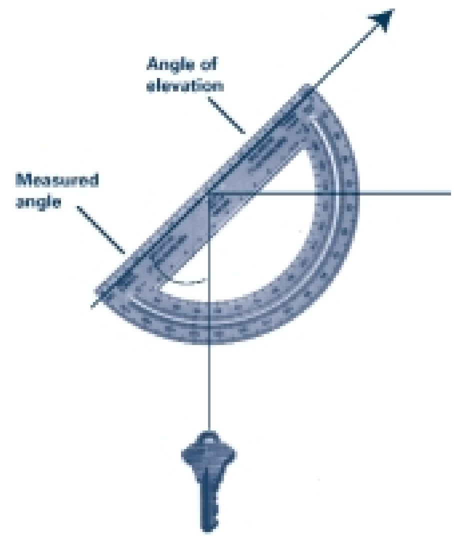
a. In the situation depicted below, what lengths and angles could you determine easily by direct measurement (and without using high-powered equipment)?



b. Which trigonometric ratios of $\angle A$ involve side BC? Of these, which also involve a measurable length?

c. Which of the trigonometric ratios of $\angle B$ involve side BC and a measurable length? If you know the size of $\angle A$, how can you find the measure of $\angle B$?

d. Krista and D'wan decided to find the height of *Bat Column* themselves. First Krista chose a spot to be point A, 20 meters from the sculpture (point C). D'wan used a *clinometer*, like the one shown at the right, to estimate the measure of $\angle A$ (the *angle of elevation* from the horizontal to the top of the bat). He measured $\angle A$ to be 55°. What is the measure of $\angle B$?



e. Krista and D'wan proceeded to find the height of the bat independently as shown below.

D'wan

I need to find BC so that

$$\frac{BC}{AC} = \tan 55°.$$

But $\tan 55° = 1.43$ *and* $AC = 20$ *m.*
So I need to solve

$$\frac{BC}{20} = 1.43.$$

If I multiply the equation by 20, I get

$$BC = 1.43 \cdot 20$$
$$BC = 28.6 \text{ m}$$

Krista

$$\tan 35° = \frac{AC}{BC}$$

$$0.7 = \frac{20}{BC}$$

Multiplying the equation by BC, I get

$$0.7 \cdot BC = 20.$$

Dividing by 0.7, I get

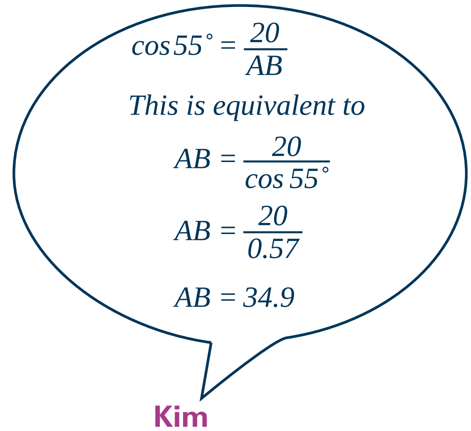
$$BC = \frac{20}{0.7} \text{ or } 28.6 \text{ m.}$$

- Analyze D'wan's thinking. Why did he multiply by 20?
- Analyze Krista's thinking. Why did she multiply by BC? Why did she divide by 0.7?
- Are the answers correct? Explain your response.

f. How could you use Krista's and D'wan's work to help estimate the height of *Bat Column*?

g. Kim said he could find the length AB (the line of sight distance) by solving $\cos 55° = \frac{AC}{AB}$. Analyze Kim's thinking shown here.

- Explain Kim's thinking.
- Is Kim correct?
- What is another way Kim could have found AB using trigonometric ratios?
- Could you find AB without using trigonometric ratios? Explain your reasoning.



3. Each part below gives data for right $\triangle ABC$. Sketch a model triangle and then, using your calculator, find the lengths of the remaining two sides.

a. $\angle B = 52°$, $a = 5$ m

b. $\angle A = 78°$, $a = 5$ mi

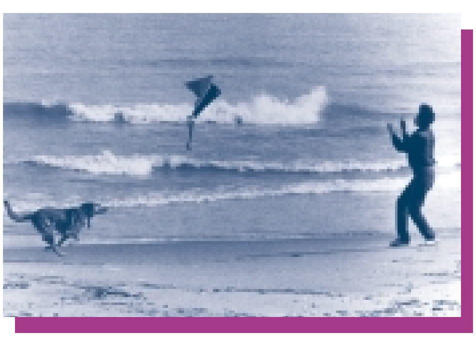
c. $\angle A = 21°$, $b = 8$ in.

d. $\angle B = 8°$, $b = 8$ ft

e. $\angle B = 37°$, $c = 42$ yd

f. $\angle A = 82°$, $c = 14$ cm

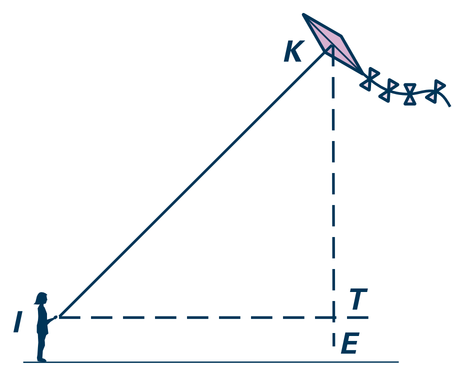
4. Terri is flying a kite and has let out 500 feet of string. Her end of the string is 3 feet off the ground.



a. If $\angle KIT$ has a measure of 40°, approximately how high off the ground is the kite?

b. As the wind picks up, Terri is able to fly the kite at a 56° angle with the horizontal. Approximately how high is the kite?

c. What is the highest Terri could fly the kite on 500 feet of string? What would be the measure of $\angle KIT$ then?



d. Experiment with your calculator to estimate the measure of $\angle KIT$ needed to fly the kite at a height of 425 feet.

In the previous situations, you used trigonometric ratios to determine an unknown or inaccessible distance. In Activity 4 Part d, you probably found a way to find the measure of an angle when you know the lengths of two sides in a right triangle.

5. Estimate (to the nearest degree) the measure of acute angle B for each of the following trigonometric ratios of $\angle B$. Check your estimate in each case by drawing a model right $\triangle ABC$, using sides whose lengths give the appropriate ratio, and then measuring $\angle B$.

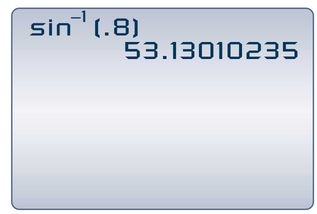
a. $\sin B = \frac{3}{5}$

b. $\cos B = \frac{1}{2}$

c. $\tan B = \frac{4}{5}$

d. Consider how you found the measure of an acute angle when you knew a trigonometric ratio for a right triangle with that angle measure. Compare your group's approach with the approaches of other groups.

6. You know how to use a calculator to produce a trigonometric ratio when you know the measure of an angle. You also can use a calculator to produce the angle when you know a trigonometric ratio as in Activity 5.



a. Suppose you know $\sin A = \frac{4}{5} = 0.8$. Use the "$\sin^{-1}$" function of your calculator to compute the angle whose sine is 0.8. (Make certain your calculator is set in degree mode.)

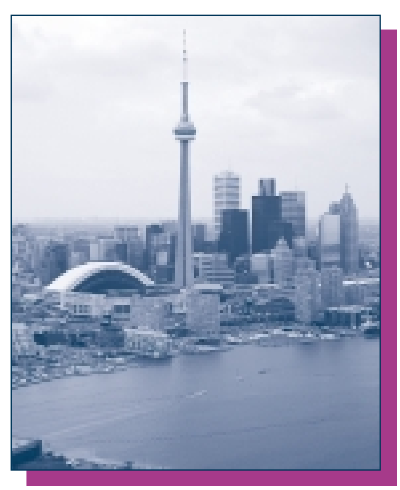
b. What would you get if you calculated the sine of the angle in the calculator display at the right?

c. Use your calculator to find the measure of $\angle B$ that corresponds to each of the ratios given in Activity 5. Compare these values to the values you obtained in that activity.

d. Use your calculator to find the measure of the angle in each of the following cases.

- $\tan B = 1.84$
- $\sin A = 0.852$
- $\cos B = 0.213$

7. The Canadian National Tower in Toronto, Ontario, is approximately 553 meters tall. This tower is the tallest free-standing structure in the world.



a. Sketch the tower and add the features described in Parts b, c, and d as you work to answer each part.

b. At some time on a sunny day, the sun makes the tower cast a 258-meter shadow. What is the measure of the angle formed by a sun ray and the ground at the tip of the shadow?

c. From the top of the Canadian National Tower, a boat is observed in Lake Ontario, approximately 8,000 meters away from the base of the tower. Assume the base of the tower is approximately level with the lake surface. What angle below the horizontal must the observer look to see the boat?

d. Estimate the line of sight distance from the observer to the boat in Part c. Find this distance using trigonometric ratios and without using them.

8. Lakeshia is about 1.7 meters tall. When standing 5 meters from her school building, her angle of sight to the top of the building is 75˚.

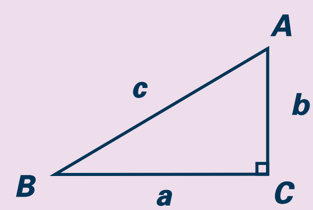
a. Estimate the height of the building.

b. Suppose Lakeshia moves to a position 10 meters from the building. What is the angle of her new line of sight to the top of the building?

c. Marcio, who is also about 1.7 meters tall, is standing on top of the building. He sees Lakeshia standing 15 meters from the building. At what angle below the horizontal is his line of sight to Lakeshia?

Checkpoint

Trigonometric ratios are useful to calculate lengths and angles in right-triangle models. Refer to the right triangle shown below in summarizing your thinking about how to use trigonometric ratios.



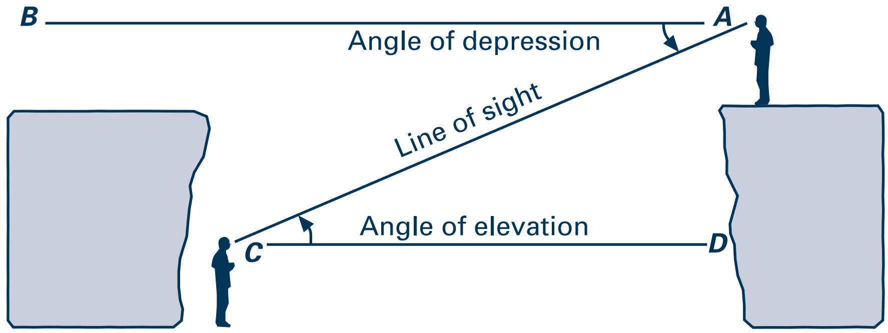
a If you knew a and the measure of $\angle B$, how would you find b? What calculator keystroke sequence would you use?

b If you knew b and c, how would you find the measure of $\angle A$? What calculator keystroke sequence would you use?

c If you knew b and the measure of $\angle B$, how would you find c? What calculator keystroke sequence would you use?

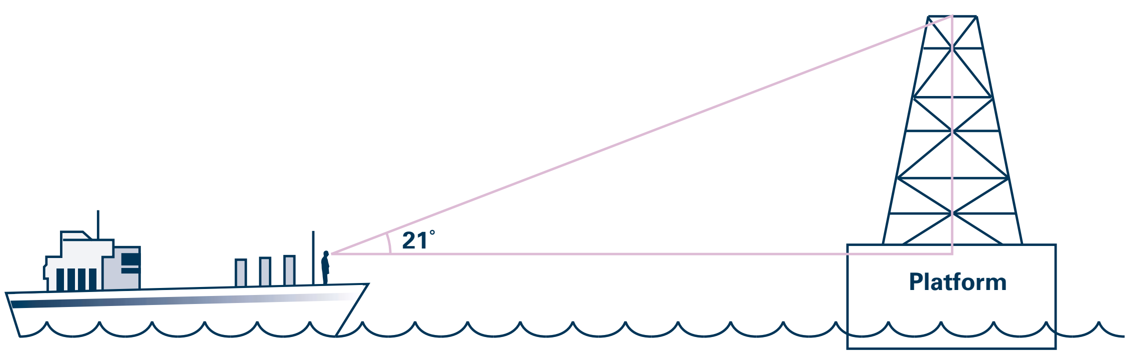
Be prepared to explain your methods to the whole class.

The **angle of elevation** to the top of an object is the angle formed by the horizontal and the line of sight to the top of the object. In the diagram below, $\angle ACD$ is the angle of elevation. The **angle of depression** to an object is the angle formed by the horizontal and the line of sight to the object below. In the diagram, $\angle BAC$ is the angle of depression.



On Your Own

A person on an oil-drilling ship in the Gulf of Mexico sees a semi-submersible platform with a tower on top of it. The tower stands 130 meters above the platform floor.



a. If the observer's position on the boat is 15 meters under the floor of the platform and the angle of elevation to the top of the rig is 21˚, what three distances can you find? Find them.

b. Suppose the boat moves so that it is 200 meters from the center of the oil rig. What is the angle of elevation now?