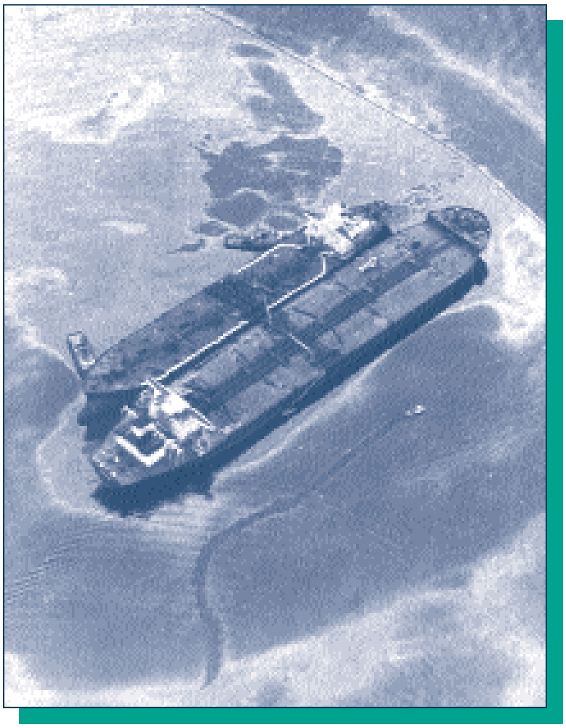
Lesson 2

Exponential Decay



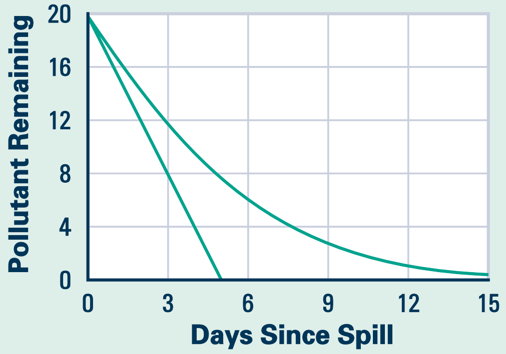
In 1989, the oil tanker Exxon Valdez ran aground in waters near the Kenai peninsula of Alaska. Over 10 million gallons of oil spread on the waters and shoreline of the area, endangering wildlife. That oil spill was eventually cleaned up—some of the oil evaporated, some was picked up by specially equipped boats, and some sank to the ocean floor as sludge.

For scientists planning environmental cleanups, it is important to be able to predict the pattern of dispersion in such contaminating spills. *Think about* the following experiment that simulates pollution of a lake or river by some poison and the cleanup.

- Mix 20 black checkers (the pollution) with 80 red checkers (the clean water).
- On the first "day" after the spill, remove 20 checkers from the mixture (without looking at the colors) and replace them with 20 red checkers (clean water). Count the number of black checkers remaining. Then shake the new mixture. This simulates a river draining off some of the polluted water and a spring or rain adding clean water to a lake.
- On the second "day" after the spill, remove 20 checkers from the new mixture (without looking at the colors) and replace them with 20 red checkers (more clean water). Count the number of black checkers remaining. Then stir the new mixture.
- Repeat the remove/replace/mix process for several more "days."

Think About This Situation

The graphs below show two possible outcomes of the pollution and cleanup simulation.



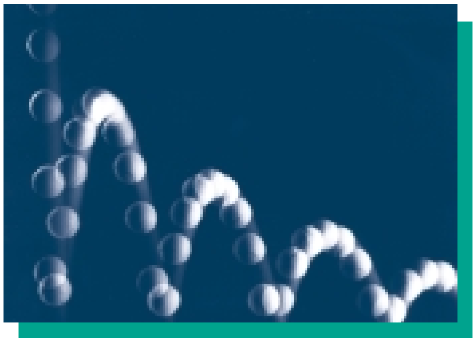
a What pattern of change is shown by each graph?

b Which graph shows the pattern of change that you would expect for this situation? Test your idea by running the experiment several times and plotting the (*time*, *pollutant remaining*) data.

c What sort of equation relating pollution P and time t would you expect to match your plot of data? Test your idea using a graphing calculator or computer.

The pollution cleanup experiment gives data in a pattern that occurs in many familiar and important problem situations. That pattern is called **exponential decay**.

INVESTIGATION 1 More Bounce to the Ounce



Most popular American sports involve balls of some sort. In playing with those balls, one of the most important factors is the bounciness or *elasticity* of the ball. For example, if a new golf ball is dropped onto a hard surface, it should rebound to about $\frac{2}{3}$ of its drop height.

Suppose a new golf ball drops downward from a height of 27 feet onto a paved parking lot and keeps bouncing up and down, again and again.

1. Make a table and plot of the data showing expected heights of the first ten bounces.

Bounce Number	0	1	2	3	4	5	6	7	8	9	10
Rebound Height	27										

a. How does the rebound height change from one bounce to the next? How is that pattern shown by the shape of the data plot?

b. What equation relating *NOW* and *NEXT* shows how to calculate the rebound height for any bounce from the height of the preceding bounce?

c. Write an equation beginning "$y = \ldots$" to model the rebound height after any number of bounces.

d. How will the data table, plot, and equations for calculating rebound height change if the ball drops first from only 15 feet?

As is the case with all mathematical models, data from actual tests of golf-ball bouncing will not match exactly the predictions from equations of ideal bounces. You can simulate the kind of quality control testing that factories do by running some experiments in your classroom.

2. Get a golf ball and a tape measure or meter stick for your group. Decide on a method for measuring the height of successive rebounds after the ball is dropped from a height of at least 8 feet. Collect data on the rebound height for successive bounces of the ball.

a. Compare the pattern of your data to that of the model that predicts rebounds which are $\frac{2}{3}$ of the drop height. Would a rebound height factor other than $\frac{2}{3}$ give a better model? Explain your reasoning.

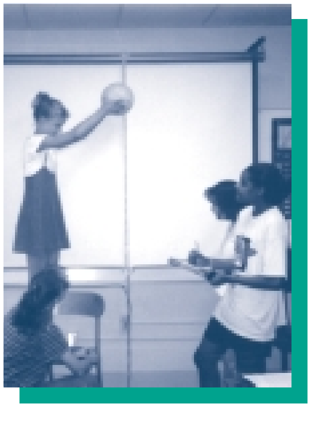
b. Write an equation using *NOW* and *NEXT* that relates the rebound height of any bounce of your tested ball to the height of the preceding bounce.

c. Write an equation beginning "$y = \ldots$" to predict the rebound height after any number of bounces.

3. Repeat the experiment of Activity 2 with some other ball such as a tennis ball or a basketball.

a. Study the data to find a reasonable estimate of the rebound height factor for your ball.

b. Write an equation using *NOW* and *NEXT* and an equation beginning "$y = \ldots$" that model the rebound height of your ball on successive bounces.



Checkpoint

Different groups might have used different balls and dropped the balls from different initial heights. However, the patterns of (*bounce number, rebound height*) data should have some similar features.

a Look back at the data from your two experiments.

- How do the rebound heights change from one bounce to the next in each case?
- How is the pattern of change in rebound height shown by the shape of the data plots in each case?

b List the equations relating *NOW* and *NEXT* and the rules ($y = \ldots$) you found for predicting the rebound heights of each ball on successive bounces.

- What do the equations relating *NOW* and *NEXT* bounce heights have in common in each case? How, if at all, are those equations different and what might be causing the differences?
- What do the rules beginning "$y = \ldots$" have in common in each case? How, if at all, are those equations different and what might be causing the differences?

c What do the tables, graphs, and equations in these examples have in common with those of the exponential growth examples in the beginning of this unit? How, if at all, are they different?

Be prepared to share and compare your data, models, and ideas with the rest of the class.

On Your Own

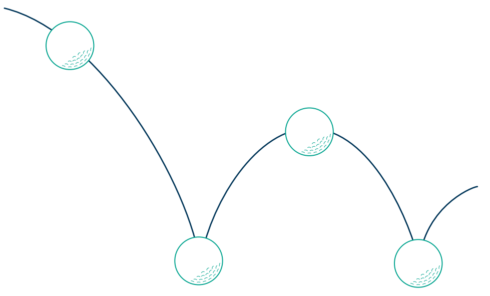
When dropped onto a hard surface, a brand new softball should rebound to about $\frac{2}{5}$ the height from which it is dropped. If a foul-tip drops straight down onto concrete after achieving a height of 25 feet, what pattern of rebound heights can be expected?

a. Make a table and plot of predicted rebound data for 5 bounces.

b. What equation relating *NOW* and *NEXT* and what rule ($y = \ldots$) giving height after any bounce match the pattern of rebound heights?

c. Here are some data from bounce tests of a softball dropped from a height of 10 feet.

Bounce Number	1	2	3	4	5
Rebound Height	3.8	1.5	0.6	0.2	0.05

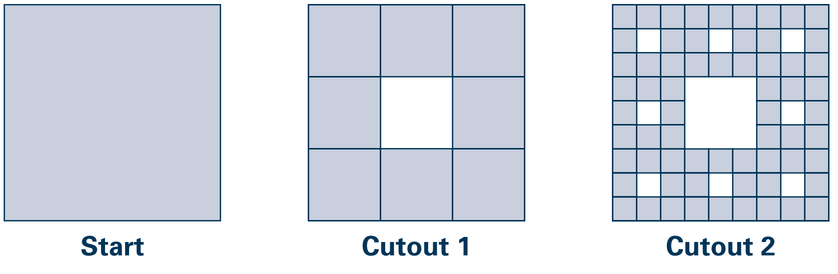


- What do these data tell you about the quality of the tested softball?
- What bounce heights would you expect from this ball if it were dropped from 20 feet instead of 10 feet?

d. What equation would model rebound height of an ideal softball if the drop were from 20 feet?

INVESTIGATION 2 Sierpinski Carpets

One of the most interesting and famous fractal patterns is named after the Polish mathematician Waclaw Sierpinski. The first two stages in forming that fractal are shown here.



Starting with a solid square "carpet" one meter on a side, smaller and smaller squares are cut out of the original in a sequence of steps. Notice how, in typical fractal style, small pieces of the design are similar to the design as a whole.

At the start of a Sierpinski carpet there is one square meter of carpet. But as cutting proceeds, there seems to be less and less carpet, and more and more hole.

1. Make a sketch showing the new holes that will appear in the third cutout from the carpet.

2. The carpet begins with an area of 1 square meter.

   a. How much of the original carpet is left after the first cutout?

   b. What fraction of the carpet left by the first cutout remains after the second cutout? How much of the original 1 square meter of carpet remains after the second cutout?

   c. What fraction of the carpet left by the second cutout remains after the third cutout? How much of the original 1 square meter of carpet remains after the third cutout?

   d. Following the pattern in the first three stages, how much of the original 1 square meter of carpet will remain after cutout 4? After cutout 5?

3. Write an equation showing the relation between the area of the remaining carpet at any stage and the next stage.

   a. What area would you predict for the carpet left after cutout 10?

   b. Find the area of the carpet left after cutout 20. After cutout 30.

4. Write an exponential equation that would allow you to calculate the area of the remaining carpet after any number of cutouts *x*, without going through all the cutouts from 1 to *x*.

   a. Make a table giving the area of the Sierpinski carpet from the start through cutout 10. Use TBLPLOT or similar software to make a plot of this data.

   b. How many cutouts are needed to get a Sierpinski carpet in which there is more hole than carpet remaining?

Checkpoint

Summarize the ways in which the table, graph, and equations for the Sierpinski carpet pattern are similar to, and different from, those for the following patterns:

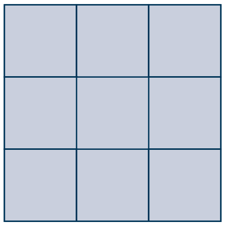
a the bouncing ball patterns of Investigation 1;

b the calling tree, king's chessboard, and bacteria growth patterns of Lesson 1.

Be prepared to share your summaries of similarities and differences with the entire class.

On Your Own

Suppose you started working on a very large Sierpinski carpet—a square that is 3 meters long on each side. Its starting area would be 9 square meters.



a. Find the area of the remaining carpet after each of the first 10 cutouts.

b. Make a plot of the (*cutout number*, *area*) data from Part a.

c. Write an equation that shows the change in area from one cutout to the next.

d. Write an exponential equation showing how to calculate the area of the carpet after any number x of cutouts.

e. How many cutouts are needed to get a Sierpinski carpet in which there is more hole than carpet remaining?

- Show how the answer to this question can be found in a table of (*cutout number*, *area*) data.
- Show how the answer to this question can be found in a plot of (*cutout number*, *area*) data.

f. How do your answers to Parts a–e compare to those for the first Sierpinski carpet with an original area of 1 square meter?

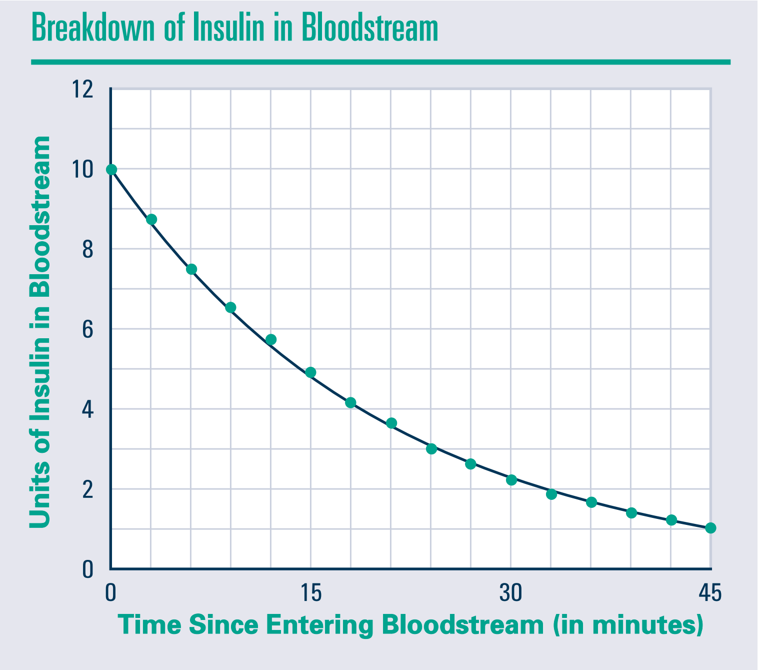
INVESTIGATION 3 Medicine and Mathematics

Drugs are a very important part of the human health equation. Many drugs are essential in preventing and curing serious physical and mental illnesses.

Diabetes, a disorder in which the body cannot metabolize glucose properly, affects people of all ages. In 1998, there were about 10 million diagnosed cases of diabetes in the United States. It was estimated that another 5 million cases remained undiagnosed.

In 5–10% of the diagnosed cases, the diabetic's body is unable to produce insulin, which is needed to process glucose.

To provide this essential hormone, these diabetics must take injections of a medicine containing insulin. The medications used (called insulin delivery systems) are designed to release insulin slowly. The insulin itself breaks down rather quickly. The rate varies greatly between individuals, but the following graph shows a typical pattern of insulin decrease.



1. Medical scientists usually are interested in the time it takes for a drug to be reduced to one half of the original dose. They call this time the **half-life** of the drug. What appears to be the half-life of insulin in this case?

2. The pattern of decay shown on this graph for insulin can be modeled well by the equation $y = 10(0.95)^x$. Experiment with your calculator or computer to see how well a table of values and graph from this rule fit the pattern in the given graph. Then explain what the values 10 and 0.95 tell about the amount of insulin in the bloodstream.

3. What equation relating *NOW* and *NEXT* shows how the amount of insulin in the blood changes from one minute to the next, once 10 units have entered the bloodstream?

4. The insulin graph shows data points for each minute following the original insulin level. But the curve connecting those points reminds us that the insulin breakdown does not occur in sudden bursts at the end of each minute! It occurs *continuously* as time passes.

What would each of the following calculations tell about the insulin decay situation? Based on the graph on the previous page, what would you expect as reasonable values for those calculations?

a. $10(0.95)^{1.5}$ **b.** $10(0.95)^{4.5}$ **c.** $10(0.95)^{18.75}$

5. Mathematicians have figured out ways to do calculations with fractional or decimal exponents so that the results fit in the pattern for whole number exponents. One of those methods is built into your graphing calculator or computer software.

a. Enter the rule Y = 10(0.95^X) in the "Y=" list of your calculator or computer software. Then complete the following table of values showing the insulin decay pattern at times other than whole minute intervals.

Time in Minutes	0	1.5	4.5	7.5	10.5	13.5	16.5	19.5
Units of Insulin in Blood	10							

b. Compare the entries in this table with data shown by points on the graph on page 446.

c. Use your rule to estimate the half-life of insulin.

Checkpoint

In this unit, you have seen that patterns of exponential change can be modeled by equations of the form $y = a(b^x)$.

a What equation relates *NOW* and *NEXT* y values of this model?

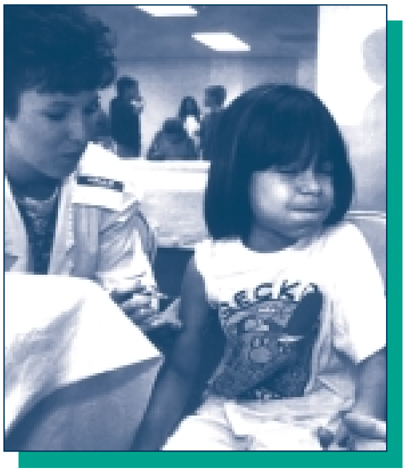
b What does the value of a tell about the situation being modeled? About the tables and graphs of (x, y) values?

c What does the value of b tell about the situation being modeled? About the tables and graphs of (x, y) values?

d How is the information provided by values of a and b in exponential equations like $y = a(b^x)$ similar to, and different from, that provided by a and b in linear equations like $y = a + bx$?

Be prepared to compare your responses with those from other groups.

On Your Own



The most famous antibiotic drug is penicillin. After its discovery in 1929, it became known as the first *miracle drug*, because it was so effective in fighting serious bacterial infections.

Drugs act somewhat differently on each person, but on average, a dose of penicillin will be broken down in the blood so that one hour after injection only 60% will remain active. Suppose a patient is given an injection of 300 milligrams of penicillin at noon.

a. Make a table showing the amount of that penicillin that will remain at hour intervals from noon until 5 PM.

b. Make a plot of the data from Part a. Explain what the pattern of that plot shows about the rate at which penicillin decays in the blood.

c. Write an equation of the form $y = a(b^x)$ that can be used to calculate the amount of penicillin remaining after any number of hours x.

d. Use the equation from Part c to produce a table showing the amount of penicillin that will remain at *quarter-hour* intervals from noon to 5 PM. What can you say about the half-life of penicillin?

e. Use the equation from Part c to graph the amount of penicillin in the blood from 0 to 10 hours. Find the time when less than 10 mg remain.